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On Hidden Symmetries of a Super Gauge Theory and Twistor String Theory

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To the memory of Gerhard Soff

Abstract

We discuss infinite-dimensional hidden symmetry algebras (and hence an infinite number of conserved nonlocal charges) of the $\mathcal{N}$-extented self-dual super Yang-Mills equations for general $\mathcal{N} \leq 4$ by using the supertwistor correspondence. Furthermore, by enhancing the supertwistor space, we construct the $\mathcal{N}$-extended self-dual super Yang-Mills hierarchies, which describe infinite sets of graded Abelian symmetries. We also show that the open topological B-model with the enhanced supertwistor space as target manifold will describe the hierarchies. Furthermore, these hierarchies will in turn – by a supersymmetric extension of Ward's conjecture – reduce to the super hierarchies of integrable models in $D < 4$ dimensions.

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1. Introduction

Within the last decades, the investigation of four-dimensional $\mathcal{N}=4$ super Yang-Mills (SYM) theory became a quite important area. In particular, the interest in this theory was again stimulated by the discovery of the AdS/CFT correspondence [1]. This conjecture states the equivalence of type IIB superstring theory on an $\mathrm{AdS}_5 \times S^5$ background with $\mathcal{N}=4$ SYM theory on $\mathbb{R}^4$.

One important tool for testing the AdS/CFT conjecture, which has been emerged lately, is integrability that appears on both sides of the correspondence. (Quantum) integrable structures in $SU(N)$ $\mathcal{N}=4$ SYM theory have first been discovered by Minahan and

Zarembo [2], inspired by the work of Berenstein, Maldacena and Nastase [3], in the large N or planar limit of the gauge theory.[2] Furthermore, it has been shown that it is possible to interpret the dilatation operator, which measures the scaling dimension of local operators, at one-loop level as Hamiltonian of an integrable quantum spin chain (see, e.g., [5] and references therein). For discussions beyond leading order see also [6]. Another development which has pointed towards integrable structures was triggered by Bena, Polchinski and Roiban [7]. Their investigation is based on the observation that the Green-Schwarz formulation of the superstring on $\mathrm{AdS}_5 \times S^5$ can be interpreted as a coset theory, where the fields take values in the supercoset space $PSU(2,2|4)/(SO(4,1) \times SO(5))$ [8]. Although this is not a symmetric space [9], they found that the classical Green-Schwarz superstring on $\mathrm{AdS}_5 \times S^5$ possesses an infinite set of conserved nonlocal charges, quite similar to those that exist in two-dimensional field theories.[3] Such charges are in turn related to Kac-Moody algebras [13,12] and generate Yangian algebras [14] as has been discussed, e.g., in [15]. For a review of Yangian algebras see, e.g., [16]. Some time later, the construction of an analogous set of nonlocal conserved charges using the pure spinor formulation of the superstring [17] on $\mathrm{AdS}_5 \times S^5$ was given in [18] (for further developments see [19]). Recently [20], it has been verified that these charges are κ-symmetric in the Green-Schwarz as well as BRST invariant in the pure spinor formulation of the superstring on $\mathrm{AdS}_5 \times S^5$. Dolan, Nappi and Witten related these nonlocal charges for the superstring to a corresponding set of nonlocal charges in the gauge theory [21] (see also [22]).

Besides the whole AdS/CFT business, the studies of $\mathcal{N} = 4$ SYM theory have recently received important input from quite another point of view. In [23], Witten has shown that perturbative $\mathcal{N} = 4$ SYM theory can be described by the D-instanton expansion of a topological string theory which turned out to be the open topological B-model whose target space is the Calabi-Yau supermanifold $\mathbb{C}P^{3|4}$. The latter space is the supersymmetric version of the twistor space [24,25]. Witten demonstrated that one can actually reproduce (initially at tree level) scattering amplitudes of the gauge theory in a simplified manner by performing string theory calculations. Since then, quite some progress in computing and understanding of such amplitudes has been made (see, e.g., [26][4]). Besides the discussion of

[2] See references [4] for an ealier account of integrable structures in QCD.

[3] These charges were also independently found by Polyakov [10]. For the construction of nonlocal conserved charges in two-dimensional sigma models see, e.g., [11,12].

[4] Cf. [27] for an earlier discussion. For properties of gravity amplitudes see, e.g., [28,29].

scattering amplitudes, a variety of other interesting aspects has been examined throughout the literature [30-41].

The open topological B-model on the supertwistor space is equivalent to holomorphic Chern-Simons (hCS) theory on the same space [42]. Therefore, the moduli space of solutions to the equations of motion of hCS theory on the supertwistor space can bijectively be mapped onto the moduli space of solutions to the equations of motion of self-dual $\mathcal{N} = 4$ SYM theory in four dimensions, as has been shown in [23] by analyzing the sheaf cohomology interpretation of the linearized field equations on the supertwistor space. This correspondence – by now known as the supertwistor correspondence – has then been pushed further beyond the linearized level in [34].

The purpose of this paper is to use this correspondence for the studies of the supertwistor construction of hidden symmetry algebras – as those mentioned above – of the $\mathcal{N}$-extended self-dual SYM equations, generalizing the results known in the literature for the bosonic self-dual Yang-Mills (SDYM) equations [43-48]. In particular, we will consider infinitesimal perturbations of the transition functions of holomorphic vector bundles over the supertwistor space. Using the Penrose-Ward transform [49][5], we relate these infinitesimal perturbations to infinitesimal symmetries of the $\mathcal{N}$-extended self-dual SYM theory, thereby obtaining infinite sets of conserved nonlocal charges. After some general words on hidden symmetry algebras, we exemplify our discussion by constructing super Kac-Moody symmetries which come from affine extensions of some Lie (super)algebra $\mathfrak{g}$. Furthermore, we consider affine extensions of the superconformal algebra to obtain super Kac-Moody-Virasoro type symmetries. Moreover, by considering a certain Abelian subalgebra of the extended superconformal algebra, we introduce a supermanifold which we call the enhanced supertwistor space. Actually, we find a whole family of such spaces which is parametrized by a set of discrete parameters. These enhanced spaces then allow us to introduce the $\mathcal{N}$-extended self-dual SYM hierarchies which describe infinite sets of graded Abelian symmetries. This generalizes the results known for the purely bosonic SDYM equations [52]. We remark that such symmetries of the latter equations are intimately connected with one-loop maximally helicity violating (MHV) amplitudes [53]. As in certain situations the enhanced supertwistor space turns out to be a Calabi-Yau space, the open topological B-model with this space as target manifold will describe certain corners of these hierarchies. Finally, we point out that since $\mathcal{N} = 3$ (4) SYM theory is related to a

[5] For reviews of twistor theory, we refer to [48,50,51].

quadric, the superambitwistor space, in $\mathbb{C}P^{3|3} \times \mathbb{C}P^{3|3}$ [25,23,34], our discussion presented below can directly be translated to the supertwistor construction of hidden symmetry algebras and thus of infinite sets of conserved nonlocal charges for the full $\mathcal{N} = 4$ SYM theory.

The paper is organized as follows. After a brief review of the supertwistor correspondence in the context of the $\mathcal{N}$-extended self-dual SYM equations in section 2, we explain in section 3 how one can construct (hidden) infinitesimal symmetries of these equations in general. In the appendix A, these symmetries are described in the context of a more mathematical language – the sheaf cohomology theory. Also in section 3, we give the construction of the above-mentioned hidden symmetry algebras. After this, we focus in section 4 on the construction of the $\mathcal{N}$-extended self-dual SYM hierarchy thereby discussing the enhanced supertwistor space. Starting from certain constraint equations, we derive the equations of motion of the hierarchies and give the superfield expansions of the accompanied fields, first in a gauge covariant manner and second in Leznov gauge (also known as light cone gauge when the Minkowski signature is chosen). The latter gauge turns out to be useful, for instance, for the geometrical interpretation of the hierarchies in the context of certain dynamical systems. Furthermore, in section 5, we describe connections between the hierarchies and the open topological B-model. After this, we present our conclusions in section 6. In the appendix B, we discuss the moduli space of almost complex structures on the superspace $\mathbb{R}^{2m|2n}$ thereby presenting yet another extension of the ordinary twistor space, which might be of interest in further investigations. Finally, the appendix C contains the superconformal algebra for general $\mathcal{N} \leq 4$.

2. Holomorphy and self-dual super Yang-Mills equations

In order to establish the formalism needed later on, we first summarize some aspects of the supertwistor correspondence between the $\mathcal{N}$-extended self-dual SYM equations on $\mathbb{R}^4$ with Euclidean signature $(++++)$ and holomorphic vector bundles over the supertwistor space

$$\mathcal{P}^{3|\mathcal{N}} \;\equiv\; \mathbb{C}P^{3|\mathcal{N}} \setminus \mathbb{C}P^{1|\mathcal{N}} \;\cong\; \mathcal{O}(1) \otimes \mathbb{C}^2 \oplus \Pi\mathcal{O}(1) \otimes \mathbb{C}^{\mathcal{N}} \;\to\; \mathbb{C}P^1, \tag{2.1}$$

where Π is the parity changing operator. We will, however, be rather brief on this and for details refer the reader to [34].

2.1. Holomorphic vector bundles and self-dual super Yang-Mills equations

The supertwistor correspondence – for the self-dual subsector[6] of the $\mathcal{N}$-extended SYM theory – is based on the double fibration

$$\mathcal{P}^{3|\mathcal{N}} \quad \stackrel{\pi_2}{\longleftarrow} \quad \mathcal{F}^{5|2\mathcal{N}} \quad \stackrel{\pi_1}{\longrightarrow} \quad \mathbb{C}_R^{4|2\mathcal{N}}, \tag{2.2}$$

where $\mathcal{P}^{3|\mathcal{N}}$ denotes the supertwistor space as given in (2.1), $\mathcal{F}^{5|2\mathcal{N}} = \mathbb{C}_R^{4|2\mathcal{N}} \times \mathbb{C}P^1$ is the correspondence space and $\mathbb{C}_R^{4|2\mathcal{N}}$ is the anti-chiral subspace of the complexification of the (super)spacetime $\mathbb{R}^{4|4\mathcal{N}}$. The mappings $\pi_{1,2}$ are projections defined momentarily. The correspondence space $\mathcal{F}^{5|2\mathcal{N}}$ can be covered by two coordinate patches, say $\{\tilde{\mathcal{U}}_+, \tilde{\mathcal{U}}_-\}$, and hence we may introduce the (complex) coordinates $(x_R^{\alpha\dot{\alpha}}, \lambda_{\dot{\alpha}}^{\pm}, \eta_i^{\dot{\alpha}})$, where $\alpha, \beta, \ldots, \dot{\alpha}, \dot{\beta}, \ldots = 1, 2$ and $i, j, \ldots = 1, \ldots, \mathcal{N}$. These coordinates are $\mathbb{Z}_2$-graded with the $x_R^{\alpha\dot{\alpha}}$, $\lambda_{\dot{\alpha}}^{\pm}$ being Graßmann even and the $\eta_i^{\dot{\alpha}}$ Graßmann odd. Moreover, the $\lambda_{\dot{\alpha}}^{\pm}$ are given by

$$(\lambda_{\dot{\alpha}}^{+}) \;\equiv\; \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} \qquad \text{and} \qquad (\lambda_{\dot{\alpha}}^{-}) \;\equiv\; \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix}, \tag{2.3}$$

with $\lambda_\pm \in \mathbb{C}P^1$, such that λ_+ is defined on U_+ and λ_- on U_-, respectively, where $\{U_+, U_-\}$ represents the canonical cover of $\mathbb{C}P^1$. The projections $\pi_{1,2}$ are defined as

$$\begin{aligned} \pi_1 \;&:\; (x_R^{\alpha\dot{\alpha}}, \lambda_{\dot{\alpha}}^{\pm}, \eta_i^{\dot{\alpha}}) \;\to\; (x_R^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}}), \\ \pi_2 \;&:\; (x_R^{\alpha\dot{\alpha}}, \lambda_{\dot{\alpha}}^{\pm}, \eta_i^{\dot{\alpha}}) \;\to\; (x_R^{\alpha\dot{\alpha}}\lambda_{\dot{\alpha}}^{\pm}, \lambda_{\dot{\alpha}}^{\pm}, \eta_i^{\dot{\alpha}}\lambda_{\dot{\alpha}}^{\pm}), \end{aligned} \tag{2.4}$$

where we call the $x_R^{\alpha\dot{\alpha}}$ anti-chiral coordinates. The supertwistor space can also be covered by two coordinate patches which we denote by $\{\mathcal{U}_+, \mathcal{U}_-\}$.[7] Note that if we are given a real structure on the supertwistor space which is induced by an anti-linear involutive automorphism τ acting on the various coordinates (and which naturally extends to arbitrary functions), the double fibration (2.2) reduces according to

$$\pi \;:\; \mathcal{P}^{3|\mathcal{N}} \;\cong\; \mathbb{R}^{4|2\mathcal{N}} \times \mathbb{C}P^1 \;\to\; \mathbb{R}^{4|2\mathcal{N}}, \tag{2.5}$$

by requiring reality of $(x_R^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}})$.[8] Here, the $\lambda_\pm$ are kept complex. One is eventually interested in real self-dual YM fields and hence it is enough to take the fibration (2.5).

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[6] Note that strictly speaking for $\mathcal{N} = 4$ it is not a subsector.

[7] Clearly, we have $U_\pm = \mathcal{U}_\pm \cap \mathbb{C}P^1$.

[8] Requiring reality of the fermionic coordinates in the case of Euclidean signature is only possible for $\mathcal{N} = 0, 2, 4$; see below.

However, it is more convenient to consider the complexification (2.2) of (2.5) first and, when necessary, to impose suitable reality conditions later on (cf. subsection 2.3). As we will also see below in our discussion of the self-dual SYM hierarchies, double fibrations are natural – even in the real setup.

We stress that from the explicit form of the projections (2.4) it follows that holomorphic sections of the bundle (2.1) are rational degree one curves, $\mathbb{C}P^1_{x_R,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$, given by the expressions

$$z^\alpha_\pm \;=\; x_R^{\alpha\dot\alpha}\lambda^\pm_{\dot\alpha} \qquad \text{and} \qquad \eta_i^\pm \;=\; \eta_i^{\dot\alpha}\lambda^\pm_{\dot\alpha}, \qquad \text{with} \qquad \lambda^\pm_{\dot\alpha} \in U_\pm, \tag{2.6}$$

and parametrized by the supermoduli $(x_R,\eta) = (x_R^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}) \in \mathbb{C}_R^{4|2\mathcal{N}}$. In this respect, the complexified (super)spacetime can be interpreted as the moduli space of rational degree one curves living in the supertwistor space. By a slight abuse of notation, we shall omit the subscript "R" on all the appearing expressions from now on.

For the discussion of the supertwistor correspondence, holomorphic vector bundles over the supertwistor space are needed. It is enough, however, to restrict ourselves to such vector bundles whose fibers are not $\mathbb{Z}_2$-graded, that is, we shall not need super vector bundles. Generally speaking, a collection consisting of five objects, $(\mathcal{E}, M, \mathrm{pr}, \mathfrak{U}, f)$, is called a holomorphic rank n vector bundle, whenever $\mathcal{E}$ and M are complex (super)manifolds, the map

$$\mathrm{pr} \,:\, \mathcal{E} \;\to\; M$$

is a surjective holomorphic projection, $\mathfrak{U} = \{\mathcal{U}_m\}$ is an open covering of M and $f = \{f_{mn}\}$ is a collection of holomorphic transition functions defined on nonempty intersections $\mathcal{U}_m \cap \mathcal{U}_n$ and being $GL(n,\mathbb{C})$-valued.

Let M be either the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ or the correspondence space $\mathcal{F}^{5|2\mathcal{N}}$. We denote the coverings of $\mathcal{P}^{3|\mathcal{N}}$ and $\mathcal{F}^{5|2\mathcal{N}}$ by $\mathfrak{U} = \{\mathcal{U}_+, \mathcal{U}_-\}$ and $\tilde{\mathfrak{U}} = \{\tilde{\mathcal{U}}_+, \tilde{\mathcal{U}}_-\}$, respectively. Consider a holomorphic vector bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ and its pull-pack $\pi_2^*\mathcal{E} \to \mathcal{F}^{5|2\mathcal{N}}$. These bundles are defined by the transition functions $f = \{f_{+-}\}$ on the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$ and $\pi_2^* f$ on $\tilde{\mathcal{U}}_+ \cap \tilde{\mathcal{U}}_-$. For notational simplicity, we shall use the same letter, f, for the transition functions of both bundles in the following course of discussion. Then, f is annihilated by the vector fields

$$D^\pm_\alpha \;=\; \lambda^{\dot\alpha}_\pm \partial_{\alpha\dot\alpha}, \qquad D^\pm_3 \;=\; \partial_{\bar\lambda_\pm} \qquad \text{and} \qquad D^i_\pm \;=\; \lambda^{\dot\alpha}_\pm \partial^i_{\dot\alpha}. \tag{2.7}$$

Here, we have abbreviated $\partial_{\alpha\dot\alpha} \equiv \partial/\partial x^{\alpha\dot\alpha}$, $\partial_{\bar\lambda_\pm} \equiv \partial/\partial\bar\lambda_\pm$, $\partial^i_{\dot\alpha} \equiv \partial/\partial\eta_i^{\dot\alpha}$. Moreover, spinor indices are raised and lowered via the ϵ-tensors, e.g., $\omega^\alpha = \epsilon^{\alpha\beta}\omega_\beta$ and $\omega^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\omega_{\dot\beta}$, where

$$(\epsilon^{\alpha\beta}) \;=\; (\epsilon^{\dot\alpha\dot\beta}) \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad (\epsilon_{\alpha\beta}) \;=\; (\epsilon_{\dot\alpha\dot\beta}) \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.8}$$

with $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta^\gamma_\alpha$ and $\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \delta^{\dot\gamma}_{\dot\alpha}$. Note that the vector fields $D^\pm_\alpha$ and $D^i_\pm$ in (2.7) are tangent to the leaves of the fibration $\pi_2 : \mathcal{F}^{5|2\mathcal{N}} \to \mathcal{P}^{3|\mathcal{N}}$.

We assume that the bundle $\mathcal{E}$ is topologically trivial and moreover holomorphically trivial when restricted to any projective line $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$. These conditions imply that there exist some smooth $GL(n,\mathbb{C})$-valued functions $\psi = \{\psi_+, \psi_-\}$, which define trivializations of $\pi_2^*\mathcal{E}$ over $\tilde{\mathcal{U}}_\pm$, such that f_{+-} can be decomposed as

$$f_{+-} \;=\; \psi_+^{-1}\psi_- \tag{2.9}$$

and

$$\partial_{\bar\lambda_\pm}\psi_\pm \;=\; 0. \tag{2.10}$$

Applying the vector fields (2.7) to (2.9), we realize that

$$\psi_+ D^+_\alpha \psi_+^{-1} \;=\; \psi_- D^+_\alpha \psi_-^{-1} \qquad \text{and} \qquad \psi_+ D^i_+ \psi_+^{-1} \;=\; \psi_- D^i_+ \psi_-^{-1}$$

must be at most linear in λ_+. Therefore, we may introduce a (Lie algebra valued) one-form $\mathcal{A}$ such that

$$D^+_\alpha \lrcorner \mathcal{A} \;\equiv\; \mathcal{A}^+_\alpha \;\equiv\; \lambda_+^{\dot\alpha}\mathcal{A}_{\alpha\dot\alpha} \;=\; \psi_\pm D^+_\alpha \psi_\pm^{-1}, \tag{2.11a}$$

$$\partial_{\bar\lambda_+} \lrcorner \mathcal{A} \;\equiv\; \mathcal{A}_{\bar\lambda_+} \;=\; 0, \tag{2.11b}$$

$$D^i_+ \lrcorner \mathcal{A} \;\equiv\; \mathcal{A}^i_+ \equiv\; \lambda_+^{\dot\alpha}\mathcal{A}^i_{\dot\alpha} \;=\; \psi_\pm D^i_+ \psi_\pm^{-1}, \tag{2.11c}$$

and hence,

$$(D^+_\alpha + \mathcal{A}^+_\alpha)\psi_\pm \;=\; 0, \tag{2.12a}$$

$$\partial_{\bar\lambda_+}\psi_\pm \;=\; 0, \tag{2.12b}$$

$$(D^i_+ + \mathcal{A}^i_+)\psi_\pm \;=\; 0. \tag{2.12c}$$

The compatibility conditions for the linear system (2.12) read as

$$\begin{gathered} [\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] + [\nabla_{\alpha\dot\beta}, \nabla_{\beta\dot\alpha}] \;=\; 0, \qquad [\nabla^i_{\dot\alpha}, \nabla_{\beta\dot\beta}] + [\nabla^i_{\dot\beta}, \nabla_{\beta\dot\alpha}] \;=\; 0, \\ \{\nabla^i_{\dot\alpha}, \nabla^j_{\dot\beta}\} + \{\nabla^i_{\dot\beta}, \nabla^j_{\dot\alpha}\} \;=\; 0, \end{gathered} \tag{2.13}$$

where we have defined the covariant derivatives

$$\nabla_{\alpha\dot\alpha} \;\equiv\; \partial_{\alpha\dot\alpha} + \mathcal{A}_{\alpha\dot\alpha} \qquad \text{and} \qquad \nabla^i_{\dot\alpha} \;\equiv\; \partial^i_{\dot\alpha} + \mathcal{A}^i_{\dot\alpha}. \tag{2.14}$$

In particular, the field content of $\mathcal{N}=4$ self-dual SYM theory consists of a self-dual gauge potential $\mathring{\mathcal{A}}_{\alpha\dot\alpha}$, four positive chirality spinors $\mathring{\chi}^i_\alpha$, six scalars $\mathring{W}^{ij} = -\mathring{W}^{ji}$, four negative chirality spinors $\mathring{\chi}_{i\dot\alpha}$ and an anti-self-dual two-form $\mathring{G}_{\dot\alpha\dot\beta}$, all in the adjoint representation of $GL(n,\mathbb{C})$. The circle refers to the lowest component in the superfield expansions of the corresponding superfields $\mathcal{A}_{\alpha\dot\alpha}$, χ^i_α, W^{ij}, $\chi_{i\dot\alpha}$ and $G_{\dot\alpha\dot\beta}$, respectively. Applying the techniques as presented in reference [54], i.e., imposing the transversal gauge,

$$\eta^{\dot\alpha}_i \mathcal{A}^i_{\dot\alpha} \;=\; 0, \tag{2.15}$$

in order to remove the superfluous gauge degrees of freedom associated with the fermionic coordinates $\eta^{\dot\alpha}_i$, we expand $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$ with respect to the odd coordinates according to

$$\begin{aligned}
\mathcal{A}_{\alpha\dot\alpha} \;&=\; \mathring{\mathcal{A}}_{\alpha\dot\alpha} + \epsilon_{\dot\alpha\dot\beta}\mathring{\chi}^i_\alpha \eta^{\dot\beta}_i + \cdots,\\
\mathcal{A}^i_{\dot\alpha} \;&=\; \epsilon_{\dot\alpha\dot\beta}\mathring{W}^{ij}\eta^{\dot\beta}_j + \tfrac{4}{3}\epsilon^{ijkl}\epsilon_{\dot\alpha\dot\beta}\mathring{\chi}_{k\dot\gamma}\eta^{\dot\gamma}_l\eta^{\dot\beta}_j \;-\\
&\quad -\tfrac{6}{5}\epsilon^{ijkl}\epsilon_{\dot\alpha\dot\beta}(\mathring{G}_{\dot\gamma\dot\delta}\delta^m_l + \epsilon_{\dot\gamma\dot\delta}[\mathring{W}^{mn},\mathring{W}_{nl}])\eta^{\dot\gamma}_k\eta^{\dot\delta}_m\eta^{\dot\beta}_j + \cdots,
\end{aligned} \tag{2.16}$$

where $\mathring{W}_{ij} \equiv \frac{1}{2}\epsilon_{ijkl}\mathring{W}^{kl}$. Therefore, (2.13) is equivalent to

$$\begin{aligned}
\mathring{f}_{\dot\alpha\dot\beta} \;&=\; 0,\\
\epsilon^{\alpha\beta}\mathring{\nabla}_{\alpha\dot\alpha}\mathring{\chi}^i_\beta \;&=\; 0,\\
\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\mathring{\nabla}_{\alpha\dot\alpha}\mathring{\nabla}_{\beta\dot\beta}\mathring{W}^{ij} + \epsilon^{\alpha\beta}\{\mathring{\chi}^i_\alpha,\mathring{\chi}^j_\beta\} \;&=\; 0,\\
\epsilon^{\dot\alpha\dot\beta}\mathring{\nabla}_{\alpha\dot\alpha}\mathring{\chi}_{i\dot\beta} - \tfrac{1}{2}\epsilon_{ijkl}[\mathring{W}^{kl},\mathring{\chi}^j_\alpha] \;&=\; 0,\\
\epsilon^{\dot\alpha\dot\beta}\mathring{\nabla}_{\alpha\dot\alpha}\mathring{G}_{\dot\beta\dot\gamma} + \{\mathring{\chi}^i_\alpha,\mathring{\chi}_{i\dot\gamma}\} + \tfrac{1}{4}\epsilon_{ijkl}[\mathring{\nabla}_{\alpha\dot\gamma}\mathring{W}^{ij},\mathring{W}^{kl}] \;&=\; 0.
\end{aligned} \tag{2.17}$$

In (2.17), $\mathring{f}_{\dot\alpha\dot\beta}$ denotes the anti-self-dual part of the curvature $[\mathring{\nabla}_{\alpha\dot\alpha},\mathring{\nabla}_{\beta\dot\beta}] = \epsilon_{\alpha\beta}\mathring{f}_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta}\mathring{f}_{\alpha\beta}$. Note that the system (2.17) takes the same form for the superfields $\mathcal{A}_{\alpha\dot\alpha}$, χ^i_α, W^{ij}, $\chi_{i\dot\alpha}$ and $G_{\dot\alpha\dot\beta}$. We remark also that the field equations for $\mathcal{N}<4$ are simply obtained by suitable truncations of (2.13) and (2.17), respectively.

Equations (2.11) can be integrated to give the formulas

$$\mathcal{A}_{\alpha\dot\alpha} \;=\; \frac{1}{2\pi\mathrm{i}}\oint_c \mathrm{d}\lambda_+ \frac{\mathcal{A}^+_\alpha}{\lambda_+\lambda^{\dot\alpha}_+} \qquad \text{and} \qquad \mathcal{A}^i_{\dot\alpha} \;=\; \frac{1}{2\pi\mathrm{i}}\oint_c \mathrm{d}\lambda_+ \frac{\mathcal{A}^i_+}{\lambda_+\lambda^{\dot\alpha}_+}, \tag{2.18}$$

where the contour $c = \{\lambda_+ \in \mathbb{C}P^1 \,|\, |\lambda_+| = 1\}$ encircles $\lambda_+ = 0$.

Note that the equations (2.11) imply that the gauge potentials $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$ do not change when we perform transformations of the form

$$\psi_\pm \ \mapsto \ \psi_\pm h_\pm,$$

where $h = \{h_+, h_-\}$ is annihilated by the vector fields (2.7). Under such transformations the transition function f_{+-} of $\pi_2^*\mathcal{E}$ transforms into a transition function $h_+^{-1} f_{+-} h_-$ of a bundle which is said to be equivalent to $\pi_2^*\mathcal{E}$,

$$f_{+-} \ \sim \ h_+^{-1} f_{+-} h_-.$$

In the following, the set of equivalence classes induced by this equivalence relation, i.e., the moduli space of holomorphic vector bundles with the properties discussed above is denoted by $\mathcal{M}_{\rm hol}(\mathcal{F}^{5|2\mathcal{N}})$.[9] Note that we have

$$\mathcal{M}_{\rm hol}(\mathcal{P}^{3|\mathcal{N}}) \ \cong \ \mathcal{M}_{\rm hol}(\mathcal{F}^{5|2\mathcal{N}}) \qquad \text{via} \qquad f \ \mapsto \ \pi_2^* f,$$

by the definition of a pull-back. Moreover, gauge transformations of the gauge potential $\mathcal{A}$ are induced by transformations of the form $\psi_\pm \mapsto g^{-1}\psi_\pm$ for some smooth $GL(n,\mathbb{C})$-valued g. Under such transformations the transition function f_{+-} does not change.

In summary, we have described a one-to-one correspondence – known as the supertwistor correspondence – between equivalence classes of holomorphic vector bundles over the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ which are topologically trivial and holomorphically trivial on the curves $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$ and gauge equivalence classes of solutions to the $\mathcal{N}$-extended self-dual SYM equations, i.e., we have the bijection

$$\mathcal{M}_{\rm hol}(\mathcal{P}^{3|\mathcal{N}}) \ \longleftrightarrow \ \mathcal{M}^{\mathcal{N}}_{\rm SDYM}, \tag{2.19}$$

where $\mathcal{M}^{\mathcal{N}}_{\rm SDYM}$ denotes the moduli space of solutions to the $\mathcal{N}$-extended self-dual SYM equations. Equations (2.18) give the explicit form of the Penrose-Ward transform.

[9] In the appendix A, we give a precise mathematical definition of this space.

2.2. Leznov gauge (light cone formalism)

Let us now consider a certain gauge, which goes under the name Leznov gauge [55]. One of the interesting issues of this (non-covariant) gauge is the fact that the self-duality equations (2.13) can be reexpressed in terms of a single Lie algebra valued chiral superfield, which we denote by Ψ. Another issue is its resemblance to the light cone gauge when the Minkowski signature is chosen.

To be explicit, assume the following expansion

$$\psi_+ \;=\; 1 + \lambda_+\Psi + \mathcal{O}(\lambda_+^2) \tag{2.20}$$

on $\tilde{\mathcal{U}}_+$ and therefore on the intersection $\tilde{\mathcal{U}}_+ \cap \tilde{\mathcal{U}}_-$. In (2.20), all λ-dependence has been made explicit. Note that generically ψ_+ is expanded according to $\psi_+ = \psi_+^{(0)} + \lambda_+\psi_+^{(1)} + \cdots$. Then, the choice $\psi_+^{(0)} = 1$ corresponds to fixing a particular gauge.[10] Upon substituting (2.20) into (2.12), we discover

$$\mathcal{A}_{\alpha\dot{1}} \;=\; \partial_{\alpha\dot{2}}\Psi, \qquad \mathcal{A}_{\alpha\dot{2}} \;=\; 0 \qquad \text{and} \qquad \mathcal{A}_{\dot{1}}^i \;=\; \partial_{\dot{2}}^i\Psi, \qquad \mathcal{A}_{\dot{2}}^i \;=\; 0. \tag{2.21}$$

The conditions (2.21) are called Leznov gauge. Plugging (2.21) into the self-duality equations (2.13), we obtain the following set of equations:[11]

$$\begin{aligned} \partial_{1\dot{1}}\partial_{2\dot{2}}\Psi - \partial_{2\dot{1}}\partial_{1\dot{2}}\Psi + [\partial_{1\dot{2}}\Psi, \partial_{2\dot{2}}\Psi] \;&=\; 0,\\ \partial_{\dot{1}}^i\partial_{\alpha\dot{2}}\Psi - \partial_{\alpha\dot{1}}\partial_{\dot{2}}^i\Psi + [\partial_{\dot{2}}^i\Psi, \partial_{\alpha\dot{2}}\Psi] \;&=\; 0,\\ \partial_{\dot{1}}^i\partial_{\dot{2}}^j\Psi + \partial_{\dot{1}}^j\partial_{\dot{2}}^i\Psi + \{\partial_{\dot{2}}^i\Psi, \partial_{\dot{2}}^j\Psi\} \;&=\; 0. \end{aligned} \tag{2.22}$$

Now it is straightforward to show that the field content, for instance, in the case of maximal $\mathcal{N}=4$ supersymmetries in Leznov gauge is given by

$$\begin{aligned} \mathcal{A}_{\alpha\dot{1}} \;&=\; \partial_{\alpha\dot{2}}\Psi \quad\Rightarrow\quad f_{\alpha\beta} \;=\; \partial_{\alpha\dot{2}}\partial_{\beta\dot{2}}\Psi,\\ \chi_\alpha^i \;&=\; \partial_{\dot{2}}^i\partial_{\alpha\dot{2}}\Psi,\\ \chi_{i\dot{2}} \;&=\; \tfrac{1}{2\cdot 3!}\epsilon_{ijkl}\partial_{\dot{2}}^j\partial_{\dot{2}}^k\partial_{\dot{2}}^l\Psi,\\ W^{ij} \;&=\; \tfrac{1}{2}\partial_{\dot{2}}^i\partial_{\dot{2}}^j\Psi,\\ G_{\dot{2}\dot{2}} \;&=\; \tfrac{1}{2\cdot 4!}\epsilon_{ijkl}\partial_{\dot{2}}^i\partial_{\dot{2}}^j\partial_{\dot{2}}^k\partial_{\dot{2}}^l\Psi, \end{aligned} \tag{2.23}$$

i.e., the superfield Ψ plays the role of a potential.

---

[10] Recall that gauge transformations are given by $\psi_+ \mapsto g^{-1}\psi_+$. Thus, the expansion (2.20) can be obtained from some general ψ_+ by performing the gauge transformation $\psi_+ \mapsto (\psi_+^{(0)})^{-1}\psi_+$.

[11] Note that the essential equation of motion is just the first one, i.e., solving the first equation to all orders in $\eta_{\dot{2}}^i$ and substituting in the remaining two, one determines the superfield Ψ to all orders in $\eta_{\dot{1}}^i$ and $\eta_{\dot{1}}^i\eta_{\dot{2}}^j$, respectively; see also [56].

2.3. Reality conditions

In the preceding discussion, we did only briefly mention the involutive automorphism τ which induces a real structure on the supertwistor space. Let us now be a bit more explicit. The Euclidean signature is related to anti-linear transformations which act on the coordinates $x^{\alpha\dot{\alpha}}$ as

$$\tau\begin{pmatrix} x^{1\dot{1}} & x^{1\dot{2}} \\ x^{2\dot{1}} & x^{2\dot{2}} \end{pmatrix} \;=\; \begin{pmatrix} \bar{x}^{2\dot{2}} & -\bar{x}^{2\dot{1}} \\ -\bar{x}^{1\dot{2}} & \bar{x}^{1\dot{1}} \end{pmatrix}, \tag{2.24}$$

where the bar denotes complex conjugation. The real subspace $\mathbb{R}^4$ of $\mathbb{C}^4$ invariant under τ is defined by the equations

$$x^{2\dot{2}} \;=\; \bar{x}^{1\dot{1}} \;\equiv\; x^4 - \mathrm{i}x^3 \qquad \text{and} \qquad x^{2\dot{1}} \;=\; -\bar{x}^{1\dot{2}} \;\equiv\; -x^2 + \mathrm{i}x^1, \tag{2.25}$$

and parametrized by real coordinates $x^\mu \in \mathbb{R}$ with $\mu = 1, \ldots, 4$. Moreover, τ acts on the $\lambda^\pm_{\dot{\alpha}}$ as

$$\tau(\lambda^+_{\dot{\alpha}}) \;=\; \begin{pmatrix} -\bar{\lambda}_+ \\ 1 \end{pmatrix} \qquad \text{and} \qquad \tau(\lambda^-_{\dot{\alpha}}) \;=\; \begin{pmatrix} -1 \\ \bar{\lambda}_- \end{pmatrix}. \tag{2.26}$$

For later convenience, we define $\tau(\lambda^\pm_{\dot{\alpha}}) \equiv \hat{\lambda}^\pm_{\dot{\alpha}}$. The action of τ on the fermionic coordinates in the case of maximal $\mathcal{N} = 4$ supersymmetries is given by

$$\tau(\eta^{\dot{\alpha}}_i) \;=\; \epsilon^{\dot{\alpha}\dot{\beta}} T_i{}^j \bar{\eta}^{\dot{\beta}}_j, \qquad \text{with} \qquad (T_i{}^j) \;\equiv\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.27}$$

Therefore, we obtain the (symplectic) Majorana condition

$$\eta^{\dot{\alpha}}_i \;=\; \epsilon^{\dot{\alpha}\dot{\beta}} T_i{}^j \bar{\eta}^{\dot{\beta}}_j. \tag{2.28}$$

Note that reality of the fermionic coordinates on Euclidean space can only be imposed if the number of supersymmetries is even [57]; the $\mathcal{N} = 0$ and $\mathcal{N} = 2$ cases are obtained by suitable truncations of the $\mathcal{N} = 4$ case (2.27) and (2.28). The action of τ on arbitrary functions is then immediate.

Finally, skew-Hermitian self-dual SYM fields can be obtained by imposing the following condition

$$f_{+-}(x, \lambda_+, \eta) \;=\; [f_{+-}(\tau(x, \lambda_+, \eta))]^\dagger \tag{2.29}$$

on the transition function. Here, the dagger denotes the extension of the complex conjugation to matrix-valued functions.

3. Holomorphy and infinitesimal symmetries

By now it is clear that we can relate – by means of the Penrose-Ward transform – holomorphic vector bundles over the supertwistor space, which are holomorphically trivial along the curves $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$ and characterized by the transition functions $f = \{f_{+-}\}$ representing $[f] \in \mathcal{M}_{\rm hol}(\mathcal{P}^{3|\mathcal{N}})$, with solutions to the $\mathcal{N}$-extended self-dual SYM equations. We can, however, associate with any open subset $\Omega \subset \mathcal{U}_+ \cap \mathcal{U}_- \subset \mathcal{P}^{3|\mathcal{N}}$ an infinite number of such $[f] \in \mathcal{M}_{\rm hol}(\mathcal{P}^{3|\mathcal{N}})$. Each class $[f]$ of equivalent holomorphic vector bundles $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ then corresponds to a class $[\mathcal{A}] \in \mathcal{M}^{\mathcal{N}}_{\rm SDYM}$ of gauge equivalent solutions to the $\mathcal{N}$-extended self-dual SYM equations. Therefore, one might wonder about the possibility of the construction of a new solution from a given one. In this section, we are going to address this issue and describe so-called infinitesimal symmetries of the $\mathcal{N}$-extended self-dual SYM equations. We thereby generalize results which have been discussed in the literature (cf., e.g., references [43-48]) for the bosonic SDYM equations. In order to discuss them, we will need to consider infinitesimal deformations of the transition functions of holomorphic vector bundles

$$\mathcal{E} \ \to \ \mathcal{P}^{3|\mathcal{N}}$$

having the above properties. By virtue of the Penrose-Ward transform, we relate them to infinitesimal perturbations $\delta\mathcal{A}$ of the gauge potential $\mathcal{A}$. Note that nontrivial infinitesimal deformations of the transition functions, denoted by δf_{+-}, correspond to vector fields on the moduli space $\mathcal{M}_{\rm hol}(\mathcal{P}^{3|\mathcal{N}})$. Nontrivial infinitesimal perturbations $\delta\mathcal{A}$ determine vector fields on the moduli space $\mathcal{M}^{\mathcal{N}}_{\rm SDYM}$.[12] Therefore, the linearized Penrose-Ward transform gives us an isomorphism

$$T_{[f]}\mathcal{M}_{\rm hol}(\mathcal{P}^{3|\mathcal{N}}) \ \cong \ T_{[\mathcal{A}]}\mathcal{M}^{\mathcal{N}}_{\rm SDYM}, \tag{3.1}$$

between the tangent spaces. We refer again to appendix A for precise mathematical definitions.

[12] More precisely, one should rather say on the solution space than on the moduli space. The latter one is obtained from the former one by factorizing with respect to the group of gauge transformations. Similar arguments hold for $(\delta f_{+-})f^{-1}_{+-}$. However, here and in the following we just ignore this subtlety.

3.1. Infinitesimal perturbations

In the following, we study small perturbations of the transition function f_{+-} of a holomorphic vector bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ and the pull-back bundle $\pi_2^*\mathcal{E} \to \mathcal{F}^{5|2\mathcal{N}}$. Any infinitesimal perturbation of f_{+-} is allowed, as small enough perturbations of the bundle $\mathcal{E}$ will – by a version of the Kodaira theorem – preserve its trivializability properties on the curves $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$. Take now $\mathcal{F}^{5|2\mathcal{N}}$ and consider an infinitesimal transformation δ of the transition function f_{+-} of the form

$$\delta \,:\, f_{+-} \;\mapsto\; \delta f_{+-} \;\equiv\; \sum_a \varepsilon_a \delta_a f_{+-}, \tag{3.2}$$

where ε_a are the infinitesimal parameters of the transformation. Generically, they are either Graßmann even (parity $p_a = 0$) or Graßmann odd (parity $p_a = 1$). Note that the total parity p_δ of δ is zero. Occasionally, we will use the Latin letters $a, b, c, \ldots$ to denote symbolically both, bosonic as well as fermionic indices. We may write

$$f_{+-} + \delta f_{+-} \;=\; (\psi_+ + \delta\psi_+)^{-1}(\psi_- + \delta\psi_-). \tag{3.3}$$

Expanding the right hand side of this equation up to first order in the perturbation, we obtain

$$\delta f_{+-} \;=\; f_{+-}\psi_-^{-1}\delta\psi_- - \psi_+^{-1}\delta\psi_+ f_{+-}. \tag{3.4}$$

Let us define the following $\mathfrak{gl}(n,\mathbb{C})$-valued function

$$\varphi_{+-} \;\equiv\; \psi_+(\delta f_{+-})\psi_-^{-1}. \tag{3.5}$$

The substitution of (3.4) into (3.5) yields

$$\varphi_{+-} \;=\; \phi_+ - \phi_-, \tag{3.6}$$

where

$$\delta\psi_\pm \;=\; -\phi_\pm\psi_\pm. \tag{3.7}$$

The $\mathfrak{gl}(n,\mathbb{C})$-valued functions ϕ_+ and ϕ_- can be extended to holomorphic functions in λ_+ and λ_- on the patches $\tilde{\mathcal{U}}_+$ and $\tilde{\mathcal{U}}_-$, respectively. We remark that finding such $\phi_\pm$ means to solve the infinitesimal variant of the Riemann-Hilbert problem. Obviously, the splitting (3.6), (3.7) and hence solutions to the Riemann-Hilbert problem are not unique, as we

certainly have the freedom to consider a new $\tilde{\phi}_\pm$ shifted by functions $\gamma_\pm$, $\tilde{\phi}_\pm = \phi_\pm + \gamma_\pm$, such that $\gamma_+ = \gamma_-$ on the intersection $\tilde{\mathcal{U}}_+ \cap \tilde{\mathcal{U}}_-$, that is, there exists some γ with $\gamma_\pm = \gamma|_{\tilde{\mathcal{U}}_\pm}$.

Equation (2.11) yields

$$\delta\mathcal{A}^+_\alpha \;=\; \delta\psi_\pm D^+_\alpha \psi_\pm^{-1} + \psi_\pm D^+_\alpha \delta\psi_\pm^{-1}, \tag{3.8a}$$

$$\delta\mathcal{A}^i_+ \;=\; \delta\psi_\pm D^i_+ \psi_\pm^{-1} + \psi_\pm D^i_+ \delta\psi_\pm^{-1}. \tag{3.8b}$$

Substituting (3.7) into these equations, we arrive at

$$\delta\mathcal{A}^+_\alpha \;=\; \nabla^+_\alpha \phi_\pm \qquad \text{and} \qquad \delta\mathcal{A}^i_+ \;=\; \nabla^i_+ \phi_\pm. \tag{3.9}$$

Here, we have introduced the definitions

$$\nabla^+_\alpha \;\equiv\; D^+_\alpha + \mathcal{A}^+_\alpha \qquad \text{and} \qquad \nabla^i_+ \;\equiv\; D^i_+ + \mathcal{A}^i_+. \tag{3.10}$$

Note that (3.10) acts adjointly in (3.9). Therefore, (3.9) together with (3.6) imply that

$$\nabla^+_\alpha \varphi_{+-} \;=\; 0 \qquad \text{and} \qquad \nabla^i_+ \varphi_{+-} \;=\; 0. \tag{3.11}$$

Moreover, the equation (2.12b) remains untouched, since $\delta\psi_\pm$ and $\phi_\pm$ are annihilated by $\partial_{\bar{\lambda}_\pm}$.

One may easily check that for the choice $\phi_\pm = \psi_\pm \chi_\pm \psi_\pm^{-1}$, where the matrix-valued functions $\chi_\pm$ depend holomorphically on the twistor coordinates $(x^{\alpha\dot\alpha}\lambda^\pm_{\dot\alpha}, \lambda^\pm_{\dot\alpha}, \eta^{\dot\alpha}_i \lambda^\pm_{\dot\alpha})$, we have $\delta\mathcal{A}_{\alpha\dot\alpha} = 0$ and $\delta\mathcal{A}^i_{\dot\alpha} = 0$, respectively. Hence, such $\phi_\pm$ define trivial perturbations. On the other hand, infinitesimal gauge transformations of the form

$$\delta\mathcal{A}_{\alpha\dot\alpha} \;=\; \nabla_{\alpha\dot\alpha}\omega \qquad \text{and} \qquad \delta\mathcal{A}^i_{\dot\alpha} \;=\; \nabla^i_{\dot\alpha}\omega,$$

where ω is some smooth $\mathfrak{gl}(n,\mathbb{C})$-valued function imply (for irreducible gauge potentials) that

$$\phi_\pm \;=\; \omega.$$

Therefore, such $\phi_\pm$ do not depend on λ_+. In particular, we have $\varphi_{+-} = 0$ and hence $\delta f_{+-} = 0$.

Finally, we obtain the formulas

$$\delta\mathcal{A}_{\alpha\dot\alpha} \;=\; \frac{1}{2\pi\mathrm{i}} \oint_c \mathrm{d}\lambda_+ \frac{\nabla^+_\alpha \phi_\pm}{\lambda_+ \lambda^{\dot\alpha}_+} \qquad \text{and} \qquad \delta\mathcal{A}^i_{\dot\alpha} \;=\; \frac{1}{2\pi\mathrm{i}} \oint_c \mathrm{d}\lambda_+ \frac{\nabla^i_+ \phi_\pm}{\lambda_+ \lambda^{\dot\alpha}_+}, \tag{3.12}$$

where the contour is again taken as $c = \{\lambda_+ \in \mathbb{C}P^1 \,|\, |\lambda_+| = 1\}$.[13]

Summarizing, to any infinitesimal deformation, $f_{+-} \mapsto f_{+-} + \delta f_{+-}$, of the transition function f_{+-} we have associated an infinitesimal symmetry transformation $\mathcal{A} \mapsto \mathcal{A} + \delta\mathcal{A}$. Clearly, by construction, the components of $\delta\mathcal{A}$ satisfy the linearized version of the $\mathcal{N}$-extended self-dual SYM equations.

3.2. An example: Hidden super Kac-Moody symmetries

After this somewhat general discussion of symmetries, let us give a first example of symmetries. Let $\mathfrak{g}$ be a some (matrix) Lie superalgebra[14] with structure constants $f_{ab}{}^c$ and (matrix) generators X_a, which satisfy

$$[X_a, X_b\} \;=\; f_{ab}{}^c X_c, \tag{3.13}$$

where $[\cdot,\cdot\}$ denotes the supercommutator

$$[A, B\} \;\equiv\; AB - (-)^{p_A p_B} BA. \tag{3.14}$$

Then, we consider the following perturbation of the transition function:

$$\delta_a^m f_{+-} \;\equiv\; \lambda_+^m [X_a, f_{+-}], \qquad \text{with} \qquad m \in \mathbb{Z}. \tag{3.15}$$

One may readily verify that the corresponding algebra is given by

$$[\delta_a^m, \delta_b^n\} \;=\; f_{ab}{}^c \delta_c^{m+n} \tag{3.16}$$

when acting on f_{+-}, i.e., we obtain the centerless super Kac-Moody algebra $\mathfrak{g}\otimes\mathbb{C}[[\lambda, \lambda^{-1}]]$. Note that in case one whishes to preserve the reality conditions, one must slightly change the above deformation. For instance, in case one chooses for $\mathfrak{g}$ the gauge algebra $\mathfrak{su}(n)$, the deformation is given according to

$$\delta_a^m f_{+-} \;=\; (\lambda_+^m + (-\lambda_-)^m)[X_a, f_{+-}].$$

[13] We see that generically the outcome of the transformation $\delta\mathcal{A}$ is a highly nonlocal expression depending on $\mathcal{A}$, i.e., we may write $\delta\mathcal{A} = F[\mathcal{A}, \partial\mathcal{A}, \ldots]$, where F is a functional whose explicit form is determined by $\phi_\pm$.

[14] The only assumption we want to made at this stage is that the corresponding Lie supergroup consists of matrices that are of c-type. See [58] for details.

We shall be continuing with our above deformation (3.15).

Next we need to find – by means of the discussion of the previous subsection – the action of δ_a^m on the components $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$ of the gauge potential $\mathcal{A}$. Consider the function φ^m_{+-a} as defined in (3.5). We obtain

$$\begin{aligned}\varphi^m_{+-a} &= \phi^m_{+a}-\phi^m_{-a}\\ &= \psi_+\lambda_+^m[X_a,f_{+-}]\psi_-^{-1} \;=\; \lambda_+^m\psi_+[X_a,\psi_+^{-1}\psi_-]\psi_-^{-1}\\ &= -\lambda_+^m[X_a,\psi_+]\psi_+^{-1}+\lambda_+^m[X_a,\psi_-]\psi_-^{-1}\\ &= \lambda_+^m\phi^0_{+a}-\lambda_+^m\phi^0_{-a},\end{aligned}$$

where in the last step we have introduced the functions $\phi^0_{\pm a}$ which are solutions of the Riemann-Hilbert problem for $m=0$,

$$\phi^0_{\pm a} \;=\; -[X_a,\psi_\pm]\psi_\pm^{-1}. \tag{3.17}$$

As the $\phi^0_{\pm a}$ are holomorphic and nonsingular in $\lambda_\pm$ on their respective domains, we expand them in powers of λ_+ on the intersection $\tilde{\mathcal{U}}_+\cap\tilde{\mathcal{U}}_-$ according to

$$\phi^0_{\pm a} \;=\; \sum_{n=0}^\infty \lambda_+^{\pm n}\phi^{0(n)}_{\pm a}, \tag{3.18}$$

where the coefficients $\phi^{0(n)}_{\pm a}$ are conserved nonlocal charges [43]. Furthermore, equations (3.12) imply

$$\begin{aligned}\delta_a^m\mathcal{A}_{\alpha\dot 1} &= \nabla_{\alpha\dot 1}\phi^{m(0)}_{+a}-\nabla_{\alpha\dot 2}\phi^{m(1)}_{+a} \;=\; \nabla_{\alpha\dot 1}\phi^{m(0)}_{-a},\\ \delta_a^m\mathcal{A}_{\alpha\dot 2} &= \nabla_{\alpha\dot 2}\phi^{m(0)}_{+a} \;=\; -\nabla_{\alpha\dot 1}\phi^{m(1)}_{-a}+\nabla_{\alpha\dot 2}\phi^{m(0)}_{-a},\end{aligned} \tag{3.19}$$

and similarly for $\mathcal{A}^i_{\dot\alpha}$.

Assume for a moment that $m\geq 0$. Then φ^m_{+-a} can be split into $\phi^m_{\pm a}$ with

$$\begin{aligned}\phi^m_{+a} &= \sum_{n=0}^\infty\lambda_+^n\phi^{m(n)}_{+a} \;=\; \sum_{n=0}^\infty\lambda_+^{m+n}\phi^{0(n)}_{+a}-\sum_{n=0}^{m-1}\lambda_+^{m-n}\phi^{0(n)}_{-a},\\ \phi^m_{-a} &= \sum_{n=0}^\infty\lambda_+^{-n}\phi^{m(n)}_{-a} \;=\; \sum_{n=0}^\infty\lambda_+^{-n}\phi^{0(m+n)}_{-a}.\end{aligned} \tag{3.20}$$

Note that for $m>0$ only ϕ^m_{-a} contains zero modes of ∂_{λ_+}. We know from our previous discussion, however, that solutions to the Riemann-Hilbert problem are not unique. So, for instance, we could have added to ϕ^m_{+a} any differentiable function which does not depend on

λ_+. But at the same time we had to add the same function to ϕ^m_{-a}, as well. However, such shifts eventually result in gauge transformations of the gauge potential $\mathcal{A}$.[15] As we are not interested in such trivial symmetries, the solution (3.20) turns out to be the appropriate choice for our further discussion. Expanding the functions $\phi^m_{\pm a}$ in powers of λ_+, we obtain the following coefficients

$$\phi^{m(n)}_{+a} \;=\; \begin{cases} \delta_{m,0}\phi^{0(0)}_{+a} & \text{for } n=0 \\ \phi^{0(n-m)}_{+a} - \phi^{0(m-n)}_{-a} & \text{for } n>0 \end{cases} \qquad \text{and} \qquad \phi^{m(n)}_{-a} \;=\; \phi^{0(m+n)}_{-a}, \tag{3.21}$$

Note that $\phi^{0(-k)}_{\pm a} = 0$ for $k > 0$. Finally, equations (3.19) together with (3.21) yield the desired transformation rules for the components of the gauge potential for $m \geq 0$

$$\begin{aligned} \delta^m_a \mathcal{A}_{\alpha\dot{1}} \;&=\; \nabla_{\alpha\dot{1}}\phi^{0(m)}_{-a} \quad \text{and} \quad \delta^m_a \mathcal{A}_{\alpha\dot{2}} \;=\; -\nabla_{\alpha\dot{1}}\phi^{0(m+1)}_{-a} + \nabla_{\alpha\dot{2}}\phi^{0(m)}_{-a}, \\ \delta^m_a \mathcal{A}^i_{\dot{1}} \;&=\; (-)^{p_a}\nabla^i_{\dot{1}}\phi^{0(m)}_{-a} \quad \text{and} \quad \delta^m_a \mathcal{A}^i_{\dot{2}} \;=\; -(-)^{p_a}\nabla^i_{\dot{1}}\phi^{0(m+1)}_{-a} + (-)^{p_a}\nabla^i_{\dot{2}}\phi^{0(m)}_{-a}. \end{aligned} \tag{3.22}$$

We stress that not all of the transformations (3.22) are nontrivial. Namely, we have

$$\delta^m_a \mathcal{A}_{\alpha\dot{2}} \;=\; 0 \;=\; \delta^m_a \mathcal{A}^i_{\dot{2}}$$

for $m > 0$, as can easily be seen from (3.19) by remembering that the coefficient $\phi^{m(0)}_{+a}$ is identically zero for all $m > 0$. This observation motivates us to assume that $\phi^{0(0)}_{+a}$ is zero, as well, implying that $\mathcal{A}_{\alpha\dot{2}}$ and $\mathcal{A}^i_{\dot{2}}$ are invariant under the action of δ^m_a for all $m \geq 0$.[16] Equation (3.17) therefore yields that $\psi^{(0)}_+$ is proportional to the identity, i.e., without loss of generality we may assume that the expansion of ψ_+ is given by (2.20). Thus, the simplification $\phi^{0(0)}_{+a} = 0$ leads us to the Leznov gauge, where the components $\mathcal{A}_{\alpha\dot{2}}$ and $\mathcal{A}^i_{\dot{2}}$ are gauge transformed to zero.

The next step is to compute the action of the commutator of two successive infinitesimal transformations, $[\delta_1, \delta_2]$, on the components of the gauge potential. In particular, we have

$$[\delta_1, \delta_2] \;=\; (-)^{p_a p_b}\epsilon^a_m \rho^b_n [\delta^m_a, \delta^n_b\},$$

[15] See also the discussion at the end of subsection 3.1.

[16] This means that all the zero modes of ∂_{λ_+} of $\varphi^m_{+-a} = \phi^m_{+a} - \phi^m_{-a}$ are contained in ϕ^m_{-a}. Alternatively, we could have changed the transformation laws to

$$\tilde{\delta}^m_a \mathcal{A}_{\alpha\dot{\alpha}} \;\equiv\; \delta^m_a \mathcal{A}_{\alpha\dot{\alpha}} - \nabla_{\alpha\dot{\alpha}}\phi^{m(0)}_{+a}$$

and similarly for $\mathcal{A}^i_{\dot{\alpha}}$ to get similar results; see below.

where ϵ^a_m and ρ^b_n are the infinitesimal parameters of the transformations δ_1 and δ_2, respectively. Explicitly, we may write

$$[\delta_1, \delta_2]\mathcal{A}_{\alpha\dot\alpha} \;=\; \delta_1(\mathcal{A}_{\alpha\dot\alpha} + \delta_2\mathcal{A}_{\alpha\dot\alpha}) - \delta_1\mathcal{A}_{\alpha\dot\alpha} - \delta_2(\mathcal{A}_{\alpha\dot\alpha} + \delta_1\mathcal{A}_{\alpha\dot\alpha}) + \delta_2\mathcal{A}_{\alpha\dot\alpha} \tag{3.23}$$

and similarly for $\mathcal{A}^i_{\dot\alpha}$. Then a straightforward calculation shows that

$$[\delta^m_a, \delta^n_b\}\mathcal{A}_{\alpha\mathrm{i}} \;=\; \nabla_{\alpha\mathrm{i}}\Sigma^{mn}_{ab} \qquad \text{and} \qquad [\delta^m_a, \delta^n_b\}\mathcal{A}^i_{\mathrm{i}} \;=\; (-)^{p_a+p_b}\nabla^i_{\mathrm{i}}\Sigma^{mn}_{ab}, \tag{3.24}$$

where

$$\Sigma^{mn}_{ab} \;\equiv\; -\delta^m_a\phi^{0(n)}_{-b} + (-)^{p_ap_b}\delta^n_b\phi^{0(m)}_{-a} - [\phi^{0(m)}_{-a}, \phi^{0(n)}_{-b}\}. \tag{3.25}$$

In this derivation we have used the coefficients (3.21). The expressions $\delta^m_a\phi^{0(n)}_{-b}$ are determined via the contour integral

$$\delta^m_a\phi^{0(n)}_{-b} \;=\; \oint_c \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}}\,\lambda_+^{n-1}\,\delta^m_a\phi^0_{-b}, \tag{3.26}$$

where the contour is taken as $c = \{\lambda_+ \in \mathbb{C}P^1 \,|\, |\lambda_+| = 1\}$. Remember that ϕ^0_{-a} is given by $\phi^0_{-a} = -[X_a, \psi_-]\psi_-^{-1}$. Hence,

$$\begin{aligned}
\delta^m_a\phi^0_{-b} \;&=\; -(-)^{p_ap_b}\left([X_b, \delta^m_a\psi_-\}\psi_-^{-1} + \phi^0_{-b}(\delta^m_a\psi_-)\psi_-^{-1}\right)\\
&=\; -(-)^{p_ap_b}\big([X_b, (\delta^m_a\psi_-)\psi_-^{-1}\} +\\
&\qquad\qquad + (-)^{p_ap_b}(\delta^m_a\psi_-)\psi_-^{-1}[X_b, \psi_-]\psi_-^{-1} + \phi^0_{-b}(\delta^m_a\psi_-)\psi_-^{-1}\big)\\
&=\; (-)^{p_ap_b}[X_b + \phi^0_{-b}, \phi^m_{-a}\}.
\end{aligned} \tag{3.27}$$

Equation (3.26) implies

$$\delta^m_a\phi^{0(n)}_{-b} \;=\; (-)^{p_ap_b}\left([X_b, \phi^{0(m+n)}_{-a}\} + \sum_{k=0}^{n}[\phi^{0(k)}_{-b}, \phi^{0(m+n-k)}_{-a}\}\right), \tag{3.28}$$

where we have used the expansions (3.21). Furthermore, we find

$$\begin{aligned}
[X_b, \phi^{0(m+n)}_{-a}\} \;&=\; -\oint_c \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}}\,\lambda_+^{m+n-1}\,[X_b, [X_a, \psi_-]\psi_-^{-1}\}\\
&=\; \oint_c \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}}\,\lambda_+^{m+n-1}\,(-)^{p_ap_b}[X_a, \psi_-]\psi_-^{-1}[X_b, \psi_-]\psi_-^{-1} -\\
&\qquad - \oint_c \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}}\,\lambda_+^{m+n-1}\,[X_b, [X_a, \psi_-]\}\psi_-^{-1}\\
&=\; (-)^{p_ap_b}\sum_{k=0}^{m+n}\phi^{0(k)}_{-a}\phi^{0(m+n-k)}_{-b} - \oint_c \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}}\,\lambda_+^{m+n-1}\,[X_b, [X_a, \psi_-]\}\psi_-^{-1}.
\end{aligned}$$

Upon using the super Jacobi identity for the triple (X_a, X_b, ψ_-), we find

$$\delta_a^m \phi_{-b}^{0(n)} - (-)^{p_a p_b} \delta_b^n \phi_{-a}^{0(m)} \;=\; -f_{ab}{}^c \phi_{-c}^{0(m+n)} + \sum_{k=0}^{m+n} [\phi_{-a}^{0(k)}, \phi_{-b}^{0(m+n-k)}\} - \\ - \sum_{k=0}^{n} [\phi_{-a}^{0(m+n-k)}, \phi_{-b}^{0(k)}\} - \sum_{k=0}^{m} [\phi_{-a}^{0(k)}, \phi_{-b}^{0(m+n-k)}\},$$

which simplifies further to

$$\delta_a^m \phi_{-b}^{0(n)} - (-)^{p_a p_b} \delta_b^n \phi_{-a}^{0(m)} \;=\; -f_{ab}{}^c \phi_{-c}^{0(m+n)} - [\phi_{-a}^{0(m)}, \phi_{-b}^{0(n)}\}. \tag{3.29}$$

Altogether, the function Σ_{ab}^{mn} as defined in (3.25) turns out to be

$$\Sigma_{ab}^{mn} \;=\; f_{ab}{}^c \phi_{-c}^{0(m+n)}. \tag{3.30}$$

Therefore, the supercommutator (3.24) is given by

$$[\delta_a^m, \delta_b^n\} \;=\; f_{ab}{}^c \delta_c^{m+n}, \tag{3.31}$$

when acting on $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}_{\dot\alpha}^i$, respectively. The superalgebra (3.31) defines the analytic half of the centerless super Kac-Moody algebra $\mathfrak{g} \otimes \mathbb{C}[[\lambda, \lambda^{-1}]]$.

So far, we have restricted to the case when $m \geq 0$. However, the line of argumentation is quite similar for negative m. In this case one may assume without loss of generality that $\phi_{-a}^{0(0)}$ is zero, which obviously implies that the other half of the components of the gauge potential are gauge transformed to zero, i.e., $\mathcal{A}_{\alpha \dot 1}$ and $\mathcal{A}_{\dot 1}^i$. Putting both cases together one eventually obtains the full affine extension of $\mathfrak{g}$, that is, the super Kac-Moody algebra $\mathfrak{g} \otimes \mathbb{C}[[\lambda, \lambda^{-1}]]$.[17]

3.3. Superconformal symmetries

In our preceding discussion, we explained how one can describe super Kac-Moody symmetries associated with a given (matrix) Lie superalgebra $\mathfrak{g}$, and which act on the solution space of the $\mathcal{N}$-extended self-dual SYM equations. In the remainder of this section we are going to push those ideas a little further by introducing affine extensions of the superconformal algebra. In doing so, we obtain an infinite number of conserved nonlocal charges, which are associated with the superconformal algebra. In the following, we shall

[17] For a similar discussion in the context of $2D$ sigma models, see, e.g., [12].

assume that we are given a real structure τ as discussed in subsection 2.3. That means, we deal with the single fibration (2.5).

It is well known that (classically) the $\mathcal{N}$-extended self-dual SYM equations are invariant under superconformal transformations. The superconformal group for $\mathcal{N} \neq 4$ is locally isomorphic to a real form of the super matrix group $SU(4|\mathcal{N})$. In the case of maximal $\mathcal{N}=4$ supersymmetries, the supergroup $SU(4|4)$ is not semi-simple and the superconformal group is considered to be a real form of the semi-simple part $PSU(4|4) \subset SU(4|4)$. The generators of the superconformal group are the translation generators $P_{\alpha\dot\alpha}$, $Q_{i\alpha}$ and $Q^i_{\dot\alpha}$, the dilatation generator D, the generators of special conformal transformations $K_{\alpha\dot\alpha}$, $K^{i\alpha}$ and $K^{\dot\alpha}_i$, the rotation generators $J_{\alpha\beta}$ and $J_{\dot\alpha\dot\beta}$, the generators T^j_i of the internal symmetry and the generator of the axial symmetry A. The latter one is absent in the case of maximal $\mathcal{N}=4$ supersymmetries. These generators can be realized in terms of the following vector fields on the anti-chiral superspace $\mathbb{R}^{4|2\mathcal{N}}$:

$$
\begin{aligned}
P_{\alpha\dot\alpha} &= \partial_{\alpha\dot\alpha}, \qquad Q_{i\alpha} \;=\; \eta^{\dot\alpha}_i\partial_{\alpha\dot\alpha}, \qquad Q^i_{\dot\alpha} \;=\; \partial^i_{\dot\alpha},\\
D &= -x^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha} - \tfrac{1}{2}\eta^{\dot\alpha}_i\partial^i_{\dot\alpha},\\
K^{\alpha\dot\alpha} &= x^{\alpha\dot\beta}(x^{\beta\dot\alpha}\partial_{\beta\dot\beta} + \eta^{\dot\alpha}_i\partial^i_{\dot\beta}),\\
K^{i\alpha} &= -x^{\alpha\dot\alpha}\partial^i_{\dot\alpha}, \qquad K^{\dot\alpha}_i \;=\; \eta^{\dot\beta}_i(x^{\alpha\dot\alpha}\partial_{\alpha\dot\beta} + \eta^{\dot\alpha}_j\partial^j_{\dot\beta}),\\
J_{\alpha\beta} &= \tfrac{1}{2}x^{\gamma\dot\alpha}\epsilon_{\gamma(\alpha}\partial_{\beta)\dot\alpha}, \qquad J_{\dot\alpha\dot\beta} \;=\; \tfrac{1}{2}\left(x^{\alpha\dot\gamma}\epsilon_{\dot\gamma(\dot\alpha}\partial_{\alpha\dot\beta)} + \eta^{\dot\gamma}_i\epsilon_{\dot\gamma(\dot\alpha}\partial^i_{\dot\beta)}\right),\\
T^j_i &= \eta^{\dot\alpha}_i\partial^j_{\dot\alpha} - \tfrac{1}{\mathcal{N}}\,\delta^j_i\eta^{\dot\alpha}_k\partial^k_{\dot\alpha}, \qquad A \;=\; \tfrac{1}{2}\eta^{\dot\alpha}_i\partial^i_{\dot\alpha},
\end{aligned}
\tag{3.32}
$$

where parentheses mean normalized symmetrization of the enclosed indices.

Infinitesimal transformations of the components $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$ of the gauge potential $\mathcal{A}$ under the action of the superconformal group are given by

$$
\delta_{N_a}\mathcal{A}_{\alpha\dot\alpha} \;=\; \mathcal{L}_{N_a}\mathcal{A}_{\alpha\dot\alpha} \qquad \text{and} \qquad \delta_{N_a}\mathcal{A}^i_{\dot\alpha} \;=\; \mathcal{L}_{N_a}\mathcal{A}^i_{\dot\alpha}, \tag{3.33}
$$

where $N_a = N^{\alpha\dot\alpha}_a\partial_{\alpha\dot\alpha} + N^{\dot\alpha}_{a\,i}\partial^i_{\dot\alpha}$ is any generator of the superconformal group, and $\mathcal{L}_{N_a}$ is the Lie superderivative along the vector field N_a. Explicitly, equations (3.33) read as

$$
\begin{aligned}
\mathcal{L}_{N_a}\mathcal{A}_{\alpha\dot\alpha} &= N_a\mathcal{A}_{\alpha\dot\alpha} + \mathcal{A}_{\beta\dot\beta}\partial_{\alpha\dot\alpha}N^{\beta\dot\beta}_a + (-)^{p_a+1}\mathcal{A}^i_{\dot\beta}\partial_{\alpha\dot\alpha}N^{\dot\beta}_{a\,i},\\
\mathcal{L}_{N_a}\mathcal{A}^i_{\dot\alpha} &= N_a\mathcal{A}^i_{\dot\alpha} + (-)^{p_a}\mathcal{A}_{\beta\dot\beta}\partial^i_{\dot\alpha}N^{\beta\dot\beta}_a + \mathcal{A}^j_{\dot\beta}\partial^i_{\dot\alpha}N^{\dot\beta}_{a\,j}.
\end{aligned}
\tag{3.34}
$$

It is not too difficult to show that for any generator N_a as given in (3.32), the transformations (3.33) together with (3.34) give a symmetry of the $\mathcal{N}$-extended self-dual SYM equations (2.13) and (2.17), respectively.

So far, we have given the action of the superconformal group on the components of the gauge potential $\mathcal{A}$. The linear system (2.12), whose compatibility conditions are the $\mathcal{N}$-extended self-dual SYM equations is, however, defined on the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$. Therefore, the question is how to define a proper action of the superconformal group on the supertwistor space such that it preserves the linear system (2.12).

Remember that the (super)twistor space describes constant complex structures on the (super)space $\mathbb{R}^{4|2\mathcal{N}}$. Thus, the action of the superconformal group on $\mathcal{P}^{3|\mathcal{N}}$ must be chosen such that it does not change a fixed constant complex structure. Consider the body $\mathbb{R}^{4|0}$ of the anti-chiral superspace $\mathbb{R}^{4|2\mathcal{N}}$. Constant complex structures on $\mathbb{R}^{4|0}$ are parametrized by the two-sphere $S^2 \cong SO(4)/U(2)$. The two-sphere S^2 can be viewed as the complex projective line $\mathbb{C}P^1$ parametrized by the coordinates $\lambda_\pm$. Then, the complex structure $\mathcal{J} = (\mathcal{J}_{\alpha\dot\alpha}{}^{\beta\dot\beta})$ on $\mathbb{R}^{4|0}$, compatible with (2.25) and (2.26), can be written as

$$\mathcal{J}_{\alpha\dot\alpha}{}^{\beta\dot\beta} \;=\; -\mathrm{i}\gamma_\pm\delta_\alpha^\beta(\lambda^\pm_{\dot\alpha}\hat\lambda^{\dot\beta}_\pm + \lambda^{\dot\beta}_\pm\hat\lambda^\pm_{\dot\alpha}), \tag{3.35}$$

where $\lambda^\pm_{\dot\alpha}$ and $\hat\lambda^\pm_{\dot\alpha}$ are as introduced in section 2 and $\gamma_\pm \equiv (1+\lambda_\pm\bar\lambda_\pm)^{-1}$. Using the explicit form (3.35) of the complex structure, one readily verifies that $\mathcal{J}_{\alpha\dot\alpha}{}^{\gamma\dot\gamma}\mathcal{J}_{\gamma\dot\gamma}{}^{\beta\dot\beta} = -\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}$.[18] On the two-sphere S^2 which parametrizes the different complex structures of $\mathbb{R}^{4|0}$, we introduce the standard complex structure $\mathfrak{J}$ which, for instance, on the U_+ patch is given by $\mathfrak{J}_{\lambda_+}{}^{\lambda_+} = \mathrm{i} = -\mathfrak{J}_{\bar\lambda_+}{}^{\bar\lambda_+}$. Thus, the complex structure on the body $\mathbb{R}^{4|0}\times\mathbb{C}P^1$ of the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ can be taken as $J = (\mathcal{J}, \mathfrak{J})$.

After introducing a complex structure on the bosonic part $\mathbb{R}^{4|0}\times\mathbb{C}P^1$ of $\mathbb{R}^{4|2\mathcal{N}}\times\mathbb{C}P^1$, we need to extend the above discussion to the full supertwistor space. In order to define a complex structure on $\mathbb{R}^{4|2\mathcal{N}}\times\mathbb{C}P^1$, recall that only an even amount of supersymmetries is possible, i.e., $\mathcal{N} = 0, 2$ or 4. Our particular choice of the symplectic Majorana condition (2.27) allows us to introduce a complex structure on the fermionic part $\mathbb{R}^{0|2\mathcal{N}}$ similar to (3.35), namely

$$\mathbf{J}^{\dot\alpha}_{i}{}^{j}_{\dot\beta} \;=\; -\mathrm{i}\gamma_\pm\delta_i^j(\lambda^\pm_{\dot\beta}\hat\lambda^{\dot\alpha}_\pm + \lambda^{\dot\alpha}_\pm\hat\lambda^\pm_{\dot\beta}). \tag{3.36}$$

Therefore, $J = (\mathcal{J}, \mathbf{J}, \mathfrak{J})$ will be the proper choice of the complex structure[19] on the supertwistor space $\mathcal{P}^{3|\mathcal{N}} \cong \mathbb{R}^{4|2\mathcal{N}}\times\mathbb{C}P^1$.

[18] We see that in the present case the corresponding Kähler two-form ω is anti-self-dual, i.e., its components are of the form $\omega_{\alpha\dot\alpha\beta\dot\beta} = \epsilon_{\alpha\beta}\mathcal{J}_{\dot\alpha\dot\beta}$. If we had chosen the ($\mathcal{N}$-extended) anti-self-dual (super) YM equations from the very beginning, the Kähler form would have been self-dual.

[19] Clearly, this choice of the complex structure does not exhaust the space of all admissible complex structures. However, in the present case it is enough to restrict ourselves to this class of complex structures. For further discussions, see appendix B.

We can now answer the initial question, namely the generators N_a given by (3.32) of the superconformal group should be lifted to vector fields $\widetilde{N}_a = \widetilde{N}_a^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha} + \widetilde{N}_{a\,i}^{\dot\alpha}\partial^i_{\dot\alpha} + \widetilde{N}_a^{\lambda_\pm}\partial_{\lambda_\pm} + \widetilde{N}_a^{\bar\lambda_\pm}\partial_{\bar\lambda_\pm}$ on the supertwistor space such that the Lie superderivative of the complex structure J along the lifted vector fields $\widetilde{N}_a$ vanishes, i.e.,

$$\mathcal{L}_{\widetilde{N}_a} J \;=\; 0. \tag{3.37}$$

Letting $\mathcal{J}_{\dot\alpha}{}^{\dot\beta} = \frac{\gamma_\pm}{2\mathrm{i}}(\lambda^\pm_{\dot\alpha}\hat\lambda^{\dot\beta}_\pm + \lambda^{\dot\beta}_\pm\hat\lambda^\pm_{\dot\alpha})$, we can write (3.37) explicitly as

$$\begin{aligned}
2\,\widetilde{N}_a\mathcal{J}_{\dot\alpha}{}^{\dot\beta} + \mathcal{J}_{\dot\gamma}{}^{\dot\alpha}\partial_{\alpha\dot\beta}\widetilde{N}_a^{\alpha\dot\gamma} - \mathcal{J}_{\dot\beta}{}^{\dot\gamma}\partial_{\alpha\dot\gamma}\widetilde{N}_a^{\alpha\dot\alpha} \;&=\; 0,\\
\mathcal{J}_{\dot\delta}{}^{\dot\gamma}\partial_{\alpha\dot\alpha}\widetilde{N}^{\dot\delta}_{a\,i} - \mathcal{J}_{\dot\alpha}{}^{\dot\beta}\partial_{\alpha\dot\beta}\widetilde{N}^{\dot\gamma}_{a\,i} \;&=\; 0,\\
\mathcal{J}_{\dot\alpha}{}^{\dot\beta}\partial^i_{\dot\gamma}\widetilde{N}_a^{\alpha\dot\alpha} - \mathcal{J}_{\dot\gamma}{}^{\dot\delta}\partial^i_{\dot\delta}\widetilde{N}_a^{\alpha\dot\beta} \;&=\; 0,\\
\delta^i_j\,\widetilde{N}_a\mathcal{J}_{\dot\alpha}{}^{\dot\beta} - (-)^{p_a}\mathcal{J}_{\dot\alpha}{}^{\dot\gamma}\partial^j_{\dot\gamma}\widetilde{N}^{\dot\beta}_{a\,i} + (-)^{p_a}\mathcal{J}_{\dot\gamma}{}^{\dot\beta}\partial^j_{\dot\beta}\widetilde{N}^{\dot\gamma}_{a\,i} \;&=\; 0,
\end{aligned} \tag{3.38}$$

whereas the equations involving $\mathfrak{J}$ tell us that the components $\widetilde{N}_a^{\lambda_\pm}$ and $\widetilde{N}_a^{\bar\lambda_\pm}$ are holomorphic in $\lambda_\pm$ and $\bar\lambda_\pm$, respectively. The final expressions for the generators (3.32) lifted to the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ satisfying (3.38) are

$$\begin{aligned}
\widetilde{P}_{\alpha\dot\alpha} \;&=\; P_{\alpha\dot\alpha}, \qquad \widetilde{Q}_{i\alpha} \;=\; Q_{i\alpha}, \qquad \widetilde{Q}^i_{\dot\alpha} \;=\; Q^i_{\dot\alpha},\\
\widetilde{D} \;&=\; D,\\
\widetilde{K}^{\alpha\dot\alpha} \;&=\; K^{\alpha\dot\alpha} + x^{\alpha\dot\beta}Z_{\dot\beta}{}^{\dot\alpha}, \qquad \widetilde{K}^{i\alpha} \;=\; K^{i\alpha}, \qquad \widetilde{K}^{\dot\alpha}_i \;=\; K^{\dot\alpha}_i + \eta^{\dot\beta}_i Z_{\dot\beta}{}^{\dot\alpha},\\
\widetilde{J}_{\alpha\beta} \;&=\; J_{\alpha\beta}, \qquad \widetilde{J}_{\dot\alpha\dot\beta} \;=\; J_{\dot\alpha\dot\beta} - \tfrac{1}{2}Z_{\dot\alpha\dot\beta},\\
\widetilde{T}^j_i \;&=\; T^j_i, \qquad \widetilde{A} \;=\; A,
\end{aligned} \tag{3.39}$$

where

$$Z_{\dot\alpha\dot\beta} \;\equiv\; \lambda^\pm_{\dot\alpha}\lambda^\pm_{\dot\beta}\partial_{\lambda_\pm} + \hat\lambda^\pm_{\dot\alpha}\hat\lambda^\pm_{\dot\beta}\partial_{\bar\lambda_\pm}. \tag{3.40}$$

Of course, these generators fullfil the same algebra.

Now we can define the infinitesimal transformation of the $GL(n,\mathbb{C})$-valued functions $\psi_\pm$ participating in (2.12) under the action of the superconformal group

$$\delta_{\widetilde{N}_a}\psi_\pm \;=\; \mathcal{L}_{\widetilde{N}_a}\psi_\pm \;=\; \widetilde{N}_a\psi_\pm, \tag{3.41}$$

where $\widetilde{N}_a$ is any of the generators given in (3.39). It is a straightforward exercise to verify explicitly that the linear system (2.12) is invariant under the transformations (3.33) and (3.41).

An example of an infinite-dimensional hidden symmetry algebra

To jump ahead of our story a bit, consider the Abelian subalgebra of the superconformal algebra which is spanned by the (super)translation generators $\widetilde{P}_{\alpha\dot{\alpha}}$ and $\widetilde{Q}^i_{\dot{\alpha}}$. On the supertwistor space we may use the coordinates $z^\alpha_\pm = x^{\alpha\dot{\alpha}}\lambda^\pm_{\dot{\alpha}}$, $\lambda^\pm_{\dot{\alpha}}$ and $\eta^\pm_i = \eta^{\dot{\alpha}}_i\lambda^\pm_{\dot{\alpha}}$. Expressing the generators $\widetilde{P}_{\alpha\dot{\alpha}}$ and $\widetilde{Q}^i_{\dot{\alpha}}$ in terms of these coordinates, we obtain

$$\widetilde{P}_{\alpha\dot{\alpha}} \;=\; \lambda^\pm_{\dot{\alpha}}\frac{\partial}{\partial z^\alpha_\pm} \qquad \text{and} \qquad \widetilde{Q}^i_{\dot{\alpha}} \;=\; \lambda^\pm_{\dot{\alpha}}\frac{\partial}{\partial \eta^\pm_i}, \tag{3.42}$$

when acting on holomorphic functions of $(z^\alpha_\pm, \lambda^\pm_{\dot{\alpha}}, \eta^\pm_i)$ on $\mathcal{P}^{3|\mathcal{N}}$. Now, consider the vector fields $\widetilde{P}^\pm_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s_\alpha}}$ and $\widetilde{Q}^i_{\pm\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i}}$ which are given by

$$\widetilde{P}^\pm_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha}} \;\equiv\; \lambda^\pm_{\dot{\alpha}_1}\cdots\lambda^\pm_{\dot{\alpha}_{2b_\alpha}}\frac{\partial}{\partial z^\alpha_\pm} \qquad \text{and} \qquad \widetilde{Q}^i_{\pm\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i}} \;\equiv\; \lambda^\pm_{\dot{\alpha}_1}\cdots\lambda^\pm_{\dot{\alpha}_{2f_i}}\frac{\partial}{\partial \eta^\pm_i}, \tag{3.43}$$

where we have introduced

$$(\lambda^+_{\dot{\alpha}_n}) \;\equiv\; \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} \quad \text{and} \quad (\lambda^-_{\dot{\alpha}_n}) \;\equiv\; \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix}, \quad \text{for} \quad n = \begin{cases} 1,\ldots,2b_\alpha \\ 1,\ldots,2f_i \end{cases}. \tag{3.44}$$

In these formulas, the parameters b_α, f_i are elements of $\{\frac{n}{2}\,|\,n \in \mathbb{N}\cup\{\infty\}\}$ whereas "b" refers to bosonic and "f" to fermionic. Clearly, the vector fields $\widetilde{P}^\pm_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha}}$ and $\widetilde{Q}^i_{\pm\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i}}$ are totally symmetric under an exchange of their dotted indices. Using (3.43), we can define the infinitesimal transformations

$$f_{+-} \;\mapsto\; \delta f_{+-}, \qquad \text{with} \qquad \delta \in \{\widetilde{P}^\pm_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha}}, \widetilde{Q}^i_{\pm\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i}}\} \tag{3.45}$$

of the transition function f_{+-}. Now we can plug δf_{+-} into (3.5) in order to obtain the functions $\phi_\pm$ which in turn give us the infinitesimal perturbations $\delta_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha}}\mathcal{A}$ and $\delta^i_{\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i}}\mathcal{A}$ of the gauge potential $\mathcal{A}$. We postpone the explicit construction of the infinitesimal transformations of the gauge potential to the next subsection.

Note that with any of the transformations

$$\delta_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha}}f_{+-}, \qquad \delta^i_{\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i}}f_{+-}, \qquad \delta_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha}}\mathcal{A}, \qquad \text{and} \qquad \delta^i_{\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i}}\mathcal{A}$$

one may associate dynamical systems on the respective moduli space and try to solve the obtained differential equations. Interestingly, integral curves of dynamical systems

on $\mathcal{M}_{\rm hol}(\mathcal{P}^{3|\mathcal{N}})$ can be described explicitly.[20] Namely, consider the following system of differential equations

$$\begin{aligned} \frac{\partial}{\partial t^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}}} f_{+-} &= \lambda^\pm_{\dot\alpha_1}\cdots\lambda^\pm_{\dot\alpha_{2b_\alpha}} \frac{\partial}{\partial z^\alpha_\pm} f_{+-}, \\ \frac{\partial}{\partial \xi_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}}} f_{+-} &= \lambda^\pm_{\dot\alpha_1}\cdots\lambda^\pm_{\dot\alpha_{2f_i}} \frac{\partial}{\partial \eta^\pm_i} f_{+-}, \end{aligned} \tag{3.46}$$

where $t^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}}$ and $\xi_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}}$ are parameters. These equations can easily be solved. The solution to (3.46) reads as

$$f_{+-} = f_{+-}(z^\alpha_\pm + t^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}}\lambda^\pm_{\dot\alpha_1}\cdots\lambda^\pm_{\dot\alpha_{2b_\alpha}}, \lambda^\pm_{\dot\alpha}, \eta^i_\pm + \xi_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}}\lambda^\pm_{\dot\alpha_1}\cdots\lambda^\pm_{\dot\alpha_{2f_i}}). \tag{3.47}$$

Note that any point of $\mathbb{R}^{4|2\mathcal{N}}$ can be obtained by a shift of the origin and hence we may put, without loss of generality, $z^\alpha_\pm$ and $\eta^i_\pm$ to zero. Therefore, (3.47) simplifies to

$$f_{+-} = f_{+-}(t^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}}\lambda^\pm_{\dot\alpha_1}\cdots\lambda^\pm_{\dot\alpha_{2b_\alpha}}, \lambda^\pm_{\dot\alpha}, \xi_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}}\lambda^\pm_{\dot\alpha_1}\cdots\lambda^\pm_{\dot\alpha_{2f_i}}), \tag{3.48}$$

where now

$$(t^{\alpha\dot 1\cdots\dot 1}, t^{\alpha\dot 2\dot 1\cdots\dot 1}, \xi_i^{\dot 1\cdots\dot 1}, \xi_i^{\dot 2\dot 1\cdots\dot 1})$$

are interpreted as coordinates on the anti-chiral superspace $\mathbb{R}^{4|2\mathcal{N}}$ whereas the others are additional moduli.

Finally, we remark that for finite sums in (3.48), $b_\alpha, f_i < \infty$, the polynomials

$$z^\alpha_\pm = t^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}}\lambda^\pm_{\dot\alpha_1}\cdots\lambda^\pm_{\dot\alpha_{2b_\alpha}} \qquad \text{and} \qquad \eta^\pm_i = \xi_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}}\lambda^\pm_{\dot\alpha_1}\cdots\lambda^\pm_{\dot\alpha_{2f_i}} \tag{3.49}$$

can be regarded as holomorphic sections of the bundle

$$\mathcal{O}(2b_1)\oplus\mathcal{O}(2b_2)\oplus\bigoplus_{i=1}^{\mathcal{N}}\Pi\mathcal{O}(2f_i) \ \to\ \mathbb{C}P^1.$$

We shall call this space – for reasons which become clear below – *enhanced supertwistor space* and denote it by $\mathcal{P}^{3|\mathcal{N}}[2b_1,2b_2|2f_1,\ldots,2f_\mathcal{N}]$. The transition functions (3.48) can then be interpreted as the transition functions of holomorphic vector bundles $\mathcal{E}\to\mathcal{P}^{3|\mathcal{N}}[2b_1,2b_2|2f_1,\ldots,2f_\mathcal{N}]$.

[20] In the next section, we will also describe a corresponding dynamical system on the moduli space $\mathcal{M}^\mathcal{N}_{\rm SDYM}$. Note again that strictly speaking we should say on the solution space.

3.4. Affine extensions of the superconformal algebra

Similar to our discussion presented in subsection 3.2, we will now focus on the construction of affine extensions of the superconformal algebra. Note that as $\mathcal{N}=3$ (4) SYM theory in four dimensions is related via the Penrose-Ward transform[21] to hCS theory on the superambitwistor space, our discussion translates, as already indicated, directly to the twistor construction of hidden symmetry algebras for the full $\mathcal{N}=4$ SYM theory.

In the previous subsection, we have explained how the generators of the superconformal group need to be lifted to the supertwistor space, such that the complex structure is not deformed. Hence, we would like to find an affine extension of the superconformal algebra which does not change the complex structure. To do this, we remember that the components $\widetilde{N}_a^{\lambda_\pm}$ and $\widetilde{N}_a^{\bar{\lambda}_\pm}$ of the generators $\widetilde{N}_a$ are holomorphic functions in $\lambda_\pm$ and $\bar{\lambda}_\pm$, respectively. Therefore, a natural choice of an extension is given by

$$\widetilde{N}_a^m \;\equiv\; \lambda_+^m \widetilde{N}_a^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \lambda_+^m \widetilde{N}^{\dot{\alpha}}_{a\,i}\partial^i_{\dot{\alpha}} + \lambda_+^m \widetilde{N}_a^{\lambda_+}\partial_{\lambda_+} + \bar{\lambda}_+^m \widetilde{N}_a^{\bar{\lambda}_+}\partial_{\bar{\lambda}_+}, \quad \text{for} \quad m\in\mathbb{Z} \tag{3.50}$$

on $\mathcal{U}_+\cap\mathcal{U}_-\subset\mathcal{P}^{3|\mathcal{N}}$. One may readily check that the Lie superderivative of the complex structure on $\mathcal{P}^{3|\mathcal{N}}$ along any $\widetilde{N}_a^m$ vanishes, thus preserving it. The Lie superbracket of two such vector fields computes to

$$[\widetilde{N}_a^m, \widetilde{N}_b^n\} \;=\; (f_{ab}{}^c + n\, g_a\, \delta_b^c - (-)^{p_a p_b} m\, g_b\, \delta_a^c)\, \widetilde{N}_c^{m+n} + K_{ab}^{mn}\, \partial_{\bar{\lambda}_+}, \tag{3.51}$$

where the $f_{ab}{}^c$ are the structure constants of the superconformal algebra (see appendix C for details) and

$$g_a \;\equiv\; \lambda_+^{-1}\widetilde{N}_a^{\lambda_+} \qquad \text{and} \qquad \bar{g}_a \;\equiv\; \bar{\lambda}_+^{-1}\widetilde{N}_a^{\bar{\lambda}_+} \tag{3.52}$$

as well as

$$\begin{aligned} K_{ab}^{mn} \;\equiv\; \Big(& n(\bar{\lambda}_+^{m+n}\bar{g}_a - \lambda_+^{m+n} g_a)\widetilde{N}_b^{\bar{\lambda}_+} - (-)^{p_a p_b} m(\bar{\lambda}_+^{m+n}\bar{g}_b - \lambda_+^{m+n} g_b)\widetilde{N}_a^{\bar{\lambda}_+} + \\ & + (\lambda_+^m\bar{\lambda}_+^n - \bar{\lambda}_+^{m+n})\widetilde{N}_a(\widetilde{N}_b^{\bar{\lambda}_+}) - (-)^{p_a p_b}(\lambda_+^n\bar{\lambda}_+^m - \bar{\lambda}_+^{m+n})\widetilde{N}_b(\widetilde{N}_a^{\bar{\lambda}_+})\Big)\,. \end{aligned} \tag{3.53}$$

Therefore, we define the following perturbation of the transition function f_{+-}

$$\delta_a^m f_{+-} \;\equiv\; \widetilde{N}_a^m f_{+-} \tag{3.54}$$

[21] See, e.g., reference [34] for details.

and hence upon action on f_{+-}, (3.51) reduces to

$$[\delta_a^m, \delta_b^n\} \;=\; (f_{ab}{}^c + n\, g_a\, \delta_b^c - (-)^{p_a p_b} m\, g_b\, \delta_a^c)\, \delta_c^{m+n}. \tag{3.55}$$

In particular, if one considers the maximal subalgebra $\mathfrak{h}$ of the superconformal algebra which consists only of those generators $\widetilde{N}_a$ that do not contain terms proportional to $Z_{\dot\alpha\dot\beta}$ given by (3.40), the algebra (3.55) simplifies further to

$$[\delta_a^m, \delta_b^n\} \;=\; h_{ab}{}^c \delta_c^{m+n}, \tag{3.56}$$

where $h_{ab}{}^c$ are the structure constants of $\mathfrak{h}$. Therefore, we obtain the centerless super Kac-Moody algebra $\mathfrak{h} \otimes \mathbb{C}[[\lambda, \lambda^{-1}]]$. Moreover, we have

$$[\delta_a^m, \delta_a^n\} \;=\; -g_a(m-n)\, \delta_a^{m+n}, \tag{3.57}$$

which resembles for $a = (\dot 1\dot 2)$, i.e., $\widetilde{N}_{\dot 1\dot 2} = \widetilde{J}_{\dot 1\dot 2}$, a centerless Virasoro algebra. In general, the algebra (3.55) can be seen as a super Kac-Moody-Virasoro type algebra.[22]

The next step is to construct the corresponding action of the δ_a^m on the components of the gauge potential and to compute their supercommutator. First, we restrict to the case when $m \geq 0$. We have already given the transformation laws of the components of the gauge potential in subsection 3.2, i.e., we can take the transformations (3.22). This time, however, the $\phi^0_{\pm a}$ are given by

$$\phi^0_{\pm a} \;=\; -(\widetilde{N}_a \psi_\pm)\psi_\pm^{-1}. \tag{3.58}$$

Again, the coefficients $\phi^{m(0)}_{+a}$ are identically zero when m is strictly positive. In order to proceed as in subsection 3.2, it is tempting to assume that $\phi^{0(0)}_{+a}$ vanishes, as well, such that only $\mathcal{A}_{\alpha\dot 1}$ and $\mathcal{A}^i_{\dot 1}$ get transformed. However, this is not possible as δ^0_a represents by construction superconformal transformations[23], and the assumption $\phi^{0(0)}_{+a} = 0$ certainly

[22] At this point we stress that for the generators $\widetilde{K}^{\alpha\dot\alpha}$ and $\widetilde{K}_i^{\dot\alpha}$ of special conformal transformations, the corresponding functions g_a also depend on $x^{\alpha\dot\alpha}$ and $\eta_i^{\dot\alpha}$ (besides $\lambda_\pm$). This will eventually lead to the exclusion of the corresponding generators δ_a^m for $m > 0$ from the representation of the symmetry algebra on the solution space of the $\mathcal{N}$-extended self-dual SYM equations; see below.

[23] Explicitly, we have $\delta^0_a \mathcal{A}_{\alpha\dot\alpha} = \mathcal{L}_{N_a} \mathcal{A}_{\alpha\dot\alpha}$ and similarly for $\mathcal{A}^i_{\dot\alpha}$; cf. subsection 3.3.

implies that $\delta^0_a\mathcal{A}_{\alpha\dot2}=0=\delta^0_a\mathcal{A}^i_{\dot2}$ in contradiction to the former statement. The best we can do is to consider instead the transformations

$$\begin{aligned}\tilde\delta^m_a\mathcal{A}_{\alpha\dot1} &\equiv \delta^m_a\mathcal{A}_{\alpha\dot1}-\nabla_{\alpha\dot1}\phi^{m(0)}_{+a} = -\nabla_{\alpha\dot2}\phi^{m(1)}_{+a} = \nabla_{\alpha\dot1}(\phi^{m(0)}_{-a}-\phi^{m(0)}_{+a}),\\ \tilde\delta^m_a\mathcal{A}_{\alpha\dot2} &\equiv \delta^m_a\mathcal{A}_{\alpha\dot2}-\nabla_{\alpha\dot2}\phi^{m(0)}_{+a} = 0\end{aligned}\tag{3.59}$$

and similarly for $\mathcal{A}^i_{\dot\alpha}$. Clearly, for $m>0$ we have $\tilde\delta^m_a=\delta^m_a$. For $m=0$, the transformations (3.59) can be interpreted as a superconformal transformation accompanied by a gauge transformation mediated by the function $\phi^{0(0)}_{+a}$. We now follow subsection 3.2 and assume without loss of generality that the power series expansion of ψ_+ is given according to (2.20). Thus, the components $\mathcal{A}_{\alpha\dot2}$ and $\mathcal{A}^i_{\dot2}$ are put to zero and we again work in Leznov gauge. Therefore, due to (3.58) the coefficient $\phi^{0(0)}_{+a}$ is identically zero only for the generators of the superconformal group for which $\widetilde N_a=N_a$. For those generators we also have

$$\mathcal{L}_{N_a}\mathcal{A}_{\alpha\dot2}\Big|_{(\mathcal{A}_{\alpha\dot2},\mathcal{A}^i_{\dot2})=(0,0)} = 0 = \mathcal{L}_{N_a}\mathcal{A}^i_{\dot2}\Big|_{(\mathcal{A}_{\alpha\dot2},\mathcal{A}^i_{\dot2})=(0,0)}.$$

We stress that by construction the transformations (3.59), when written in Leznov gauge, do preserve this gauge.[24]

Next, we need to compute the commutator $[\tilde\delta_1,\tilde\delta_2]$ of two successive transformations. We do this computation in two steps: first, we consider the case where $m,n>0$ and second the case where $m>0$ and $n=0$.[25]

Assume that $m,n>0$. In this case, we can use the equations (3.24) together with (3.25). We simply need to replace δ^m_a by $\tilde\delta^m_a$. Remember that for $m>0$ we have $\tilde\delta^m_a\mathcal{A}_{\alpha\dot1}=\nabla_{\alpha\dot1}\phi^{0(m)}_{-a}$ and $\tilde\delta^m_a\mathcal{A}^i_{\dot1}=(-)^{p_a}\nabla^i_{\dot1}\phi^{0(m)}_{-a}$. Therefore, similar to (3.27) the expression $\tilde\delta^m_a\phi^0_{-b}$ is given by

$$\tilde\delta^m_a\phi^0_{-b} = (-)^{p_ap_b}[\widetilde N_b+\phi^0_{-b},\phi^m_{-a}\} = (-)^{p_ap_b}\left(\widetilde N_b\phi^m_{-a}+[\phi^0_{-b},\phi^m_{-a}\}\right),\tag{3.60}$$

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[24] Note that the only nonvanishing Lie superderivatives (in Leznov gauge) are

$$\begin{aligned}&\mathcal{L}_{K^{\alpha\dot2}}\mathcal{A}_{\beta\dot2} = \mathcal{A}_{\beta\dot1}x^{\alpha\dot1}, \quad \mathcal{L}_{K^{\dot2}_i}\mathcal{A}_{\alpha\dot2} = \mathcal{A}_{\alpha\dot1}\eta^{\dot1}_i, \quad \mathcal{L}_{J_{\dot1\dot1}}\mathcal{A}_{\beta\dot2} = \tfrac12\mathcal{A}_{\alpha\dot1},\\ &\mathcal{L}_{K^{\alpha\dot2}}\mathcal{A}^i_{\dot2} = \mathcal{A}^i_{\dot1}x^{\alpha\dot1}, \quad \mathcal{L}_{K^{\dot2}_i}\mathcal{A}^j_{\dot2} = -\mathcal{A}^j_{\dot1}\eta^{\dot1}_i, \quad \mathcal{L}_{J_{\dot1\dot1}}\mathcal{A}^i_{\dot2} = \tfrac12\mathcal{A}^i_{\dot1}.\end{aligned}$$

Computing the expressions $\partial_{\alpha\dot2}\phi^{0(0)}_{+a}$ and $\partial^i_{\dot2}\phi^{0(0)}_{+a}$ for those generators, one realizes that $\tilde\delta^m_a\mathcal{A}_{\alpha\dot2}=0=\tilde\delta^m_a\mathcal{A}^i_{\dot2}$ is indeed true.

[25] The case $m=n=0$ is obvious.

which directly follows from $\phi^0_{-b} = -(\widetilde{N}_b\psi_-)\psi_-^{-1}$, and hence

$$\begin{aligned}
\tilde{\delta}^m_a\phi^{0(n)}_{-b} \ &= \ (-)^{p_ap_b}\oint_c \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}}\,\lambda_+^{n-1}\,\left(\widetilde{N}_b\phi^m_{-a} + [\phi^0_{-b},\phi^m_{-a}\}\right)\\
&= \ (-)^{p_ap_b}\Bigg(N_b\phi^{0(m+n)}_{-a} - \sum_{k=-1}^{1}\,(n+k)\,g_b^{(k)}\phi^{0(m+n+k)}_{-a} + \\
&\qquad\qquad + \sum_{k=0}^{n}\,[\phi^{0(k)}_{-b},\phi^{0(m+n-k)}_{-a}\}\Bigg),
\end{aligned} \tag{3.61}$$

where the coefficients $g_a^{(k)}$ are those of the function g_a defined in (3.52) when expanded in powers of λ_+. Moreover, we have

$$\begin{aligned}
N_b\phi^{0(m+n)}_{-a} \ &= \ -\oint_c \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}}\,\lambda_+^{m+n-1}\,N_b((\widetilde{N}_a\psi_-)\psi_-^{-1})\\
&= \ -\oint_c \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}}\,\lambda_+^{m+n-1}\,\left(\widetilde{N}_b((\widetilde{N}_a\psi_-)\psi_-^{-1}) - \widetilde{N}_b^{\lambda_+}\partial_{\lambda_+}((\widetilde{N}_a\psi_-)\psi_-^{-1})\right)\\
&= \ -\oint_c \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}}\,\lambda_+^{m+n-1}\,(\widetilde{N}_b\widetilde{N}_a\psi_-)\psi_-^{-1} + (-)^{p_ap_b}\sum_{k=0}^{m+n}\,\phi^{0(k)}_{-a}\phi^{0(m+n-k)}_{-b} + \\
&\qquad + \sum_{k=-1}^{1}\,(m+n+k)\,g_b^{(k)}\phi^{0(m+n+k)}_{-a}.
\end{aligned}$$

Putting everything together, we obtain

$$\begin{aligned}
\tilde{\delta}^m_a\phi^{0(n)}_{-b} - (-)^{p_ap_b}\tilde{\delta}^n_b\phi^{0(m)}_{-a} \ &= \ -f_{ab}{}^c\phi^{0(m+n)}_{-c} - [\phi^{0(m)}_{-a},\phi^{0(n)}_{-b}\} - \\
&\quad - \sum_{k=-1}^{1}\,\left(n\,g_a^{(k)}\delta^c_b - (-)^{p_ap_b}m\,g_b^{(k)}\delta^c_a\right)\phi^{0(m+n+k)}_{-c}.
\end{aligned} \tag{3.62}$$

Therefore, by virtue of the transformation laws (3.59), we conclude that the algebra does only close, when all the $g_a^{(k)}$ are constants. From (3.39) we realize that we need to exclude the generators $\widetilde{K}^{\alpha\dot{\alpha}}$ and $\widetilde{K}_i^{\dot{\alpha}}$. In the following, let $\mathfrak{h}$ be the maximal subalgebra of the superconformal algebra which does not contain $\widetilde{K}^{\alpha\dot{\alpha}}$ and $\widetilde{K}_i^{\dot{\alpha}}$.[26] Hence, equation (3.62) implies

$$[\tilde{\delta}^m_a,\tilde{\delta}^n_b\} \ = \ h_{ab}{}^c\tilde{\delta}^{m+n}_c + \sum_{k=-1}^{1}\,\left(n\,g_a^{(k)}\delta^c_b - (-)^{p_ap_b}m\,g_b^{(k)}\delta^c_a\right)\tilde{\delta}^{m+n+k}_c, \tag{3.63}$$

[26] See also appendix C.

for $m,n>0$. In (3.63), the $h_{ab}{}^c$ are the structure constants of $\mathfrak{h}$.

It remains to compute $[\tilde\delta_a^m,\tilde\delta_b^0\}$ for $m>0$. A straightforward calculation shows that the function Σ_{ab}^{m0} entering the formulas (3.24) is given by

$$\begin{aligned}\Sigma_{ab}^{m0} &= -\tilde\delta_a^m(\phi_{-b}^{0(0)}-\phi_{+b}^{0(0)})+(-)^{p_ap_b}\tilde\delta_b^0\phi_{-a}^{0(m)}-[\phi_{-a}^{0(m)},\phi_{-b}^{0(0)}-\phi_{+b}^{0(0)}\}\\ &= -\tilde\delta_a^m\phi_{-b}^{0(0)}+(-)^{p_ap_b}\tilde\delta_b^0\phi_{-a}^{0(m)}-[\phi_{-a}^{0(m)},\phi_{-b}^{0(0)}\}+[\phi_{-a}^{0(m)},\phi_{+b}^{0(0)}\}+\tilde\delta_a^m\phi_{+b}^{0(0)}.\end{aligned} \tag{3.64}$$

Recall that for $m=0$ the transformation $\tilde\delta_b^0$ consists of a superconformal transformation accompanied by a gauge transformation mediated by the function $\phi_{+b}^{0(0)}$, that is, $\tilde\delta_b^0=\delta_b^0+\delta_b^g$. Therefore, from varying $\phi_{-a}^0=-(\widetilde N_a\psi_-)\psi_-^{-1}$, we obtain[27]

$$\tilde\delta_b^0\phi_{-a}^0 \;=\; (-)^{p_ap_b}[\widetilde N_a+\phi_{-a}^0,\phi_{-b}^0-\phi_{+b}^{0(0)}\}. \tag{3.65}$$

Thus, the sum over the first four terms of (3.64) gives

$$\begin{aligned}&\tilde\delta_a^m\phi_{-b}^{0(0)}-(-)^{p_ap_b}\tilde\delta_b^0\phi_{-a}^{0(m)}+[\phi_{-a}^{0(m)},\phi_{-b}^{0(0)}-\phi_{+b}^{0(0)}\} \;=\\ &\qquad = -f_{ab}{}^c\phi_{-c}^{0(m)}+(-)^{p_ap_b}\sum_{k=-1}^{1} m\,g_b^{(k)}\phi_{-a}^{0(m+k)}-(-)^{p_ap_b}g_b^{(-1)}\phi_{-a}^{0(m-1)}.\end{aligned} \tag{3.66}$$

Furthermore, in a similar manner we find

$$\begin{aligned}\tilde\delta_a^m\phi_{+b}^{0(0)} &= (-)^{p_ap_b}\oint_c\frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}}\,\lambda_+^{-1}\left(\widetilde N_b\phi_{+a}^m+[\phi_{+b}^0,\phi_{+a}^m\}\right)\\ &= (-)^{p_ap_b}g_b^{(-1)}\phi_{+a}^{m(1)} \;=\; g_b^{(-1)}(-)^{p_ap_b}\left(\phi_{+a}^{0(1-m)}-\phi_{-a}^{0(m-1)}\right),\end{aligned} \tag{3.67}$$

where in the last step we have used (3.21). Therefore, we end up with

$$\Sigma_{ab}^{m0} \;=\; f_{ab}{}^c\phi_{-c}^{0(m)}-(-)^{p_ap_b}\sum_{k=-1}^{1} m\,g_b^{(k)}\phi_{-a}^{0(m+k)}+(-)^{p_ap_b}g_b^{(-1)}\phi_{+a}^{0(1-m)} \tag{3.68}$$

for $m>0$. Assuming again that the coefficients $g_a^{(k)}$ are independent of $x^{\alpha\dot\alpha}$ and $\eta_i^{\dot\alpha}$, we arrive at

$$[\tilde\delta_a^m,\tilde\delta_b^0\}\mathcal{A}_{\alpha\mathrm{i}} \;=\; h_{ab}{}^c\tilde\delta_c^m\mathcal{A}_{\alpha\mathrm{i}}-(-)^{p_ap_b}\sum_{k=-1}^{1} m\,g_b^{(k)}\tilde\delta_a^{m+k}\mathcal{A}_{\alpha\mathrm{i}} \tag{3.69}$$

and similarly for $\mathcal{A}_{\mathrm{i}}^i$.

In summary, we have obtained the algebra

$$[\tilde\delta_a^m,\tilde\delta_b^n\} \;=\; h_{ab}{}^c\tilde\delta_c^{m+n}+\sum_{k=-1}^{1}\left(n\,g_a^{(k)}\delta_b^c-(-)^{p_ap_b}m\,g_b^{(k)}\delta_a^c\right)\tilde\delta_c^{m+n+k} \tag{3.70}$$

for $m,n\geq 0$. The computations for negative m,n go along similar lines and one eventually finds (as in subsection 3.2) the algebra (3.70) for all $m,n\in\mathbb{Z}$.

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[27] Generally, we could have written $\tilde\delta_b^n\phi_{-a}^0=(-)^{p_ap_b}[\widetilde N_a+\phi_{-a}^0,\phi_{-b}^n-\phi_{+b}^{n(0)}\}$ from the very beginning, since for $n>0$ we have $\phi_{+b}^{n(0)}=0$.

Some remarks

At first sight it seems a little surprising that we had to exclude the generators of special conformal transformations $\widetilde{K}^{\alpha\dot{\alpha}}$ and $\widetilde{K}^{\dot{\alpha}}_i$ and hence to restrict to the subalgebra $\mathfrak{h}$. However, remember that we work on the open subset $\mathcal{P}^{3|\mathcal{N}} = \mathbb{C}P^{3|\mathcal{N}} \setminus \mathbb{C}P^{1|\mathcal{N}}$ of $\mathbb{C}P^{3|\mathcal{N}}$. The latter space is the compactification of $\mathcal{P}^{3|\mathcal{N}}$ which corresponds via the supertwistor correspondence to compactified superspacetime, i.e., one relates holomorphic vector bundles over $\mathbb{C}P^{3|\mathcal{N}}$ via the Penrose-Ward transform to the $\mathcal{N}$-extended self-dual SYM theory on compactified spacetime.[28] Recall that $\mathcal{P}^{3|\mathcal{N}}$ is covered by two coordinate patches, while the space $\mathbb{C}P^{3|\mathcal{N}}$ by four. Special conformal transformations relate different coordinate patches and hence by removing the subspace $\mathbb{C}P^{1|\mathcal{N}}$ one does not have as much freedom as one has for $\mathbb{C}P^{3|\mathcal{N}}$. Moreover, special conformal transformations transform $\lambda_\pm$ into a function which depends on $x^{\alpha\dot{\alpha}}$, $\eta^{\dot{\alpha}}_i$ and $\lambda_\pm$. Thus, they do not preserve the fibration $\mathcal{P}^{3|\mathcal{N}} \to \mathbb{C}P^1$. Therefore, in order to extend the full superconformal symmetry algebra one should rather work on the compactified supertwistor space $\mathbb{C}P^{3|\mathcal{N}}$. Then, however, the linear system (2.12) needs to be changed to incorporate the Levi-Civita superconnection which is induced from compactified superspacetime.

4. Holomorphy and self-dual super Yang-Mills hierarchies

In the previous section, we have introduced infinitesimal symmetries which were related with translational symmetries on $\mathbb{R}^{4|2\mathcal{N}}$. Moreover, we discussed the dynamical system (3.46). By solving this system, we discovered the space

$$\mathcal{P}^{3|\mathcal{N}}[2b_1, 2b_2|2f_1, \ldots, 2f_\mathcal{N}] \;\equiv\; \mathcal{O}(2b_1) \oplus \mathcal{O}(2b_2) \oplus \bigoplus_{i=1}^{\mathcal{N}} \Pi\mathcal{O}(2f_i) \;\to\; \mathbb{C}P^1, \qquad (4.1)$$

which we called the enhanced supertwistor space. Note that this space can be viewed as an open subset of the weighted projective space

$$W\mathbb{C}P^{3|\mathcal{N}}[2b_1, 2b_2, 1, 1|2f_1, \ldots, 2f_\mathcal{N}]$$

by removing the subspace $W\mathbb{C}P^{1|\mathcal{N}}[2b_1, 2b_2|2f_1, \ldots, 2f_\mathcal{N}]$.[29] The goal of this section is to explore the space $\mathcal{P}^{3|\mathcal{N}}[2b_1, 2b_2|2f_1, \ldots, 2f_\mathcal{N}]$ in more detail and furthermore to discuss

[28] For Euclidean signature one has S^4, for Minkowski $S^3 \times S^1$ and for Kleinian $S^2 \times S^2$, respectively.

[29] For related discussions on weighted projective spaces in the context of the supertwistor correspondence see references [23,35,36].

holomorphic vector bundles over it. We will eventually obtain the *$\mathcal{N}$-extended self-dual SYM hierarchy.* We remark that for $b_\alpha = f_i = \frac{1}{2}$, the subsequent discussion reduces, of course, to the one presented in section 2. If no confusion arises, we will write, for brevity, $\mathcal{P}^{3|\mathcal{N}}_{b,f}$ instead of $\mathcal{P}^{3|\mathcal{N}}[2b_1, 2b_2|2f_1, \ldots, 2f_\mathcal{N}]$.

4.1. Enhanced supertwistor space

In the following, we again work in the complex setup. As before, one can impose proper reality conditions afterwards (cf. subsection 2.3).

Obviously, since the enhanced supertwistor space $\mathcal{P}^{3|\mathcal{N}}_{b,f}$ is fibered over the Riemann sphere[30], it can be covered by two coordinate patches which we denote – as before – by $\mathfrak{U} = \{\mathcal{U}_+, \mathcal{U}_-\}$. Moreover, we introduce the following coordinates on $\mathcal{U}_\pm$:

$$(z^\alpha_\pm, \lambda_\pm, \eta^\pm_i), \quad \text{with} \quad z^\alpha_+ \;=\; \lambda_+^{2b_\alpha}\, z^\alpha_-, \quad \lambda_+ \;=\; \lambda_+^2 \lambda_- \quad \text{and} \quad \eta^+_i \;=\; \lambda_+^{2f_i}\, \eta^-_i . \tag{4.2}$$

According to the preceding discussion, we are interested in holomorphic sections of the bundle (4.1). Contrary to the supertwistor space, these are rational curves of degree $(2b_1, 2b_2; 2f_1, \ldots, 2f_\mathcal{N})$, $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}_{b,f}$, which are given by the expressions

$$z^\alpha_\pm \;=\; x^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}} \lambda^\pm_{\dot\alpha_1} \cdots \lambda^\pm_{\dot\alpha_{2b_\alpha}} \quad \text{and} \quad \eta^\pm_i \;=\; \eta_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}} \lambda^\pm_{\dot\alpha_1} \cdots \lambda^\pm_{\dot\alpha_{2f_i}} \quad \text{on} \quad \mathcal{U}_\pm, \tag{4.3}$$

and parametrized by the supermoduli

$$(x, \eta) \;=\; (x^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}}, \eta_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}}) \;\in\; \mathbb{C}^{2(b_1+b_2+1)|2(f_1+\cdots+f_\mathcal{N})+\mathcal{N}}.$$

Therefore, we have enhanced the moduli space $\mathbb{C}^{4|2\mathcal{N}}$ of rational curves of degree one[31] living in the supertwistor space to the moduli space $\mathbb{C}^{2(b_1+b_2+1)|2(f_1+\cdots+f_\mathcal{N})+\mathcal{N}}$ of rational degree $(2b_1, 2b_2; 2f_1, \ldots, 2f_\mathcal{N})$ curves sitting inside the enhanced supertwistor space. We remark that for the choices for which at most one of the dotted indices is equal to two, the coordinates $(x^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}}, \eta_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}})$ might be interpreted – up to some unimportant prefactors entering through permutations of their dotted indices – as coordinates on the anti-chiral superspace $\mathbb{C}^{4|2\mathcal{N}}$.[32] Therefore, the complexified (super)spacetime is naturally embedded in the space $\mathbb{C}^{2(b_1+b_2+1)|2(f_1+\cdots+f_\mathcal{N})+\mathcal{N}}$ and we may write

$$\mathbb{C}^{2(b_1+b_2+1)|2(f_1+\cdots+f_\mathcal{N})+\mathcal{N}} \;\cong\; \mathbb{C}^{4|2\mathcal{N}} \oplus \mathbb{C}^{2(b_1+b_2-1)|2(f_1+\cdots+f_\mathcal{N})-\mathcal{N}}.$$

[30] That is, it is a vector bundle over $\mathbb{C}P^1$.

[31] More precisely, we should write of degree $(1, 1; 1, \ldots, 1)$.

[32] Cf. subsection 3.2.

Altogether, the equations (4.3) allow us to introduce the double fibration[33]

$$\mathcal{P}^{3|\mathcal{N}}_{b,f} \quad \xleftarrow{\pi_2} \quad \mathcal{F}^{2(b_1+b_2)+3|2(f_1+\cdots+f_\mathcal{N})+\mathcal{N}} \quad \xrightarrow{\pi_1} \quad \mathbb{C}^{2(b_1+b_2+1)|2(f_1+\cdots+f_\mathcal{N})+\mathcal{N}}, \tag{4.4}$$

where now the correspondence space is given by the direct product

$$\mathcal{F}^{2(b_1+b_2)+3|2(f_1+\cdots+f_\mathcal{N})+\mathcal{N}} \;=\; \mathbb{C}^{2(b_1+b_2+1)|2(f_1+\cdots+f_\mathcal{N})+\mathcal{N}} \times \mathbb{C}P^1.$$

Furthermore, the natural choice of coordinates on $\mathcal{F}^{2(b_1+b_2)+3|2(f_1+\cdots+f_\mathcal{N})+\mathcal{N}}$ reads as

$$(x^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}}, \lambda^\pm_{\dot\alpha}, \eta_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}}) \tag{4.5}$$

and hence the action of projections π_1 and π_2 in the fibration (4.4) is

$$\begin{aligned} \pi_1 \,:\, (x^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}}, \lambda^\pm_{\dot\alpha}, \eta_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}}) &\to (x^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}}, \eta_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}}),\\ \pi_2 \,:\, (x^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}}, \lambda^\pm_{\dot\alpha}, \eta_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}}) &\to (x^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}}\lambda^\pm_{\dot\alpha_1}\cdots\lambda^\pm_{\dot\alpha_{2b_\alpha}}, \lambda^\pm_{\dot\alpha}, \eta_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}}\lambda^\pm_{\dot\alpha_1}\cdots\lambda^\pm_{\dot\alpha_{2f_i}}), \end{aligned} \tag{4.6}$$

where

$$(\lambda^+_{\dot\alpha_n}) \;\equiv\; \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} \quad \text{and} \quad (\lambda^-_{\dot\alpha_n}) \;\equiv\; \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix}, \quad \text{for} \quad n = \begin{cases} 1,\ldots,2b_\alpha \\ 1,\ldots,2f_i \end{cases}, \tag{4.7}$$

as before.

In summary, we learn that a point in $\mathbb{C}^{2(b_1+b_2+1)|2(f_1+\cdots+f_\mathcal{N})+\mathcal{N}}$ corresponds to a projective line $\mathbb{C}P^1_{x,\eta}$ in $\mathcal{P}^{3|\mathcal{N}}_{b,f}$ given by solutions to (4.3) for fixed $(x^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}}, \eta_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}})$. Conversely, a point in $\mathcal{P}^{3|\mathcal{N}}_{b,f}$ corresponds to an affine subspace (strictly speaking an affine β-subspace) in $\mathbb{C}^{2(b_1+b_2+1)|2(f_1+\cdots+f_\mathcal{N})+\mathcal{N}}$ of dimension $(2(b_1+b_2)|2(f_1+\cdots+f_\mathcal{N}))$ which is defined by solutions to (4.3) for fixed $(z^\alpha_\pm, \lambda_\pm, \eta_i^\pm)$.

4.2. Self-dual super Yang-Mills hierarchies

In subsection 4.1, we have introduced the enhanced supertwistor space $\mathcal{P}^{3|\mathcal{N}}_{b,f}$ and the double fibration (4.4). Now it is natural to consider holomorphic vector bundles over $\mathcal{P}^{3|\mathcal{N}}_{b,f}$ and to ask for the output of the Penrose-Ward transform. Eventually, we will obtain the promised $\mathcal{N}$-extended self-dual SYM hierarchies.

[33] Note that in the real setup this double fibration does not reduce to a single fibration like (2.5).

Let $\mathcal{E}$ be a holomorphic vector bundle over $\mathcal{P}^{3|\mathcal{N}}_{b,f}$ and $\pi_2^*\mathcal{E}$ the pull-back of $\mathcal{E}$ to the correspondence space $\mathcal{F}^{2(b_1+b_2)+3|2(f_1+\cdots+f_{\mathcal{N}})+\mathcal{N}}$. The covering of the latter is denoted by $\tilde{\mathfrak{U}} = \{\tilde{\mathcal{U}}_+, \tilde{\mathcal{U}}_-\}$. These bundles are defined by transition functions[34] $f = \{f_{+-}\}$ which are annihilated by the vector fields

$$\begin{aligned} D^{\pm}_{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha-1}} &= \lambda^{\dot\alpha}_{\pm}\partial_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha-1}}, \qquad D^{\pm}_{3} = \partial_{\bar\lambda_\pm}, \\ D^{i}_{\pm\dot\alpha_1\cdots\dot\alpha_{2f_i-1}} &= \lambda^{\dot\alpha}_{\pm}\partial^{i}_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f_i-1}}. \end{aligned} \tag{4.8}$$

In these expressions, we have introduced the abbreviations

$$\partial_{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}} \equiv \frac{\partial}{\partial x^{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha}}} \qquad \text{and} \qquad \partial^{i}_{\dot\alpha_1\cdots\dot\alpha_{2f_i}} \equiv \frac{\partial}{\partial\eta_i^{\dot\alpha_1\cdots\dot\alpha_{2f_i}}}.$$

We note that the vector fields $D^{\pm}_{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha-1}}$ and $D^{i}_{\pm\dot\alpha_1\cdots\dot\alpha_{2f_i-1}}$ in (4.8) form a basis of the tangent spaces of the $(2(b_1+b_2)|2(f_1+\cdots+f_{\mathcal{N}}))$-dimensional leaves of the fibration $\pi_2 : \mathcal{F}^{2(b_1+b_2)+3|2(f_1+\cdots+f_{\mathcal{N}})+\mathcal{N}} \to \mathcal{P}^{3|\mathcal{N}}_{b,f}$.

The requirement of topological triviality of the bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}_{b,f}$ allows us to split the transition function f_{+-} according to

$$f_{+-} = \psi_+^{-1}\psi_-, \tag{4.9}$$

whereas holomorphic triviality of $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}_{b,f}$ along the rational curves $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}_{b,f}$ ensures that there exist $\psi_\pm$ such that $\partial_{\bar\lambda_\pm}\psi_\pm = 0$. Proceeding as in section 2, we write

$$\mathcal{A}^{+}_{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha-1}} \equiv \lambda^{\dot\alpha}_{+}\mathcal{A}_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha-1}} = \psi_\pm D^{+}_{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha-1}}\psi_\pm^{-1}, \tag{4.10a}$$

$$\mathcal{A}_{\bar\lambda_+} = 0, \tag{4.10b}$$

$$\mathcal{A}^{i}_{+\dot\alpha_1\cdots\dot\alpha_{2f_i-1}} \equiv \lambda^{\dot\alpha}_{+}\mathcal{A}^{i}_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f_i-1}} = \psi_\pm D^{i}_{+\dot\alpha_1\cdots\dot\alpha_{2f_i-1}}\psi_\pm^{-1}, \tag{4.10c}$$

and therefore

$$(D^{+}_{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha-1}} + \mathcal{A}^{+}_{\alpha\dot\alpha_1\cdots\dot\alpha_{2b_\alpha-1}})\psi_\pm = 0, \tag{4.11a}$$

$$\partial_{\bar\lambda_+}\psi_\pm = 0, \tag{4.11b}$$

$$(D^{i}_{+\dot\alpha_1\cdots\dot\alpha_{2f_i-1}} + \mathcal{A}^{i}_{+\dot\alpha_1\cdots\dot\alpha_{2f_i-1}})\psi_\pm = 0. \tag{4.11c}$$

[34] Here, we again use the same letter f for both bundles.

We note that for $b_\alpha = f_i = \frac{1}{2}$ this system reduces, of course, to the old one given by (2.12). Moreover, we have the following symmetry properties

$$\mathcal{A}_{\alpha\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha-1}} = \mathcal{A}_{\alpha\dot{\alpha}(\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha-1})} \quad \text{and} \quad \mathcal{A}^i_{\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1}} = \mathcal{A}^i_{\dot{\alpha}(\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1})}. \tag{4.12}$$

The compatibility conditions for (4.11) read as

$$\begin{aligned}
[\nabla_{\alpha\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha-1}}, \nabla_{\beta\dot{\beta}\dot{\beta}_1\cdots\dot{\beta}_{2b_\beta-1}}] + [\nabla_{\alpha\dot{\beta}\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha-1}}, \nabla_{\beta\dot{\alpha}\dot{\beta}_1\cdots\dot{\beta}_{2b_\beta-1}}] &= 0,\\
[\nabla^i_{\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1}}, \nabla_{\beta\dot{\beta}\dot{\beta}_1\cdots\dot{\beta}_{2b_\beta-1}}] + [\nabla^i_{\dot{\beta}\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1}}, \nabla_{\beta\dot{\alpha}\dot{\beta}_1\cdots\dot{\beta}_{2b_\beta-1}}] &= 0,\\
\{\nabla^i_{\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1}}, \nabla^j_{\dot{\beta}\dot{\beta}_1\cdots\dot{\beta}_{2f_j-1}}\} + \{\nabla^i_{\dot{\beta}\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1}}, \nabla^j_{\dot{\alpha}\dot{\beta}_1\cdots\dot{\beta}_{2f_j-1}}\} &= 0.
\end{aligned} \tag{4.13}$$

Here, we have defined

$$\begin{aligned}
\nabla_{\alpha\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha-1}} &\equiv \partial_{\alpha\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha-1}} + \mathcal{A}_{\alpha\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha-1}},\\
\nabla^i_{\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1}} &\equiv \partial^i_{\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1}} + \mathcal{A}^i_{\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1}}.
\end{aligned} \tag{4.14}$$

We remark that the components

$$\mathcal{A}_{\alpha\dot{\alpha}\dot{1}\cdots\dot{1}} \qquad \text{and} \qquad \mathcal{A}^i_{\dot{\alpha}\dot{1}\cdots\dot{1}}$$

coincide with the components $\mathcal{A}_{\alpha\dot{\alpha}}$ and $\mathcal{A}^i_{\dot{\alpha}}$ of the gauge potential on $\mathbb{C}^{4|\mathcal{N}}$ we have introduced in (2.11). In the sequel, we shall refer to (4.13) as the *truncated $\mathcal{N}$-extended self-dual SYM hierarchy*. The *full hierarchy* are then obtained by taking the limit $b_\alpha, f_i \to \infty$. In this asymptotic regime, we define (symbolically) the space

$$\mathcal{P}^{3|\mathcal{N}}_\infty \equiv \lim_{b_\alpha, f_i\to\infty} \mathcal{P}^{3|\mathcal{N}}_{b,f}, \tag{4.15}$$

which we call the *fully enhanced supertwistor space*.

From the equations (4.10) it follows that

$$\begin{aligned}
\mathcal{A}_{\alpha\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha-1}} &= \frac{1}{2\pi\mathrm{i}} \oint_c \mathrm{d}\lambda_+ \frac{\mathcal{A}^+_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha-1}}}{\lambda_+\lambda_+^{\dot{\alpha}}},\\
\mathcal{A}^i_{\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1}} &= \frac{1}{2\pi\mathrm{i}} \oint_c \mathrm{d}\lambda_+ \frac{\mathcal{A}^i_{+\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1}}}{\lambda_+\lambda_+^{\dot{\alpha}}},
\end{aligned} \tag{4.16}$$

where $c = \{\lambda_+ \in \mathbb{C}P^1 \,|\, |\lambda_+| = 1\}$. As in the previous discussion, the equations (4.16) make the Penrose-Ward transform explicit.

In summary, we have extended the one-to-one correspondence between equivalence classes of holomorphic vector bundles over the supertwistor space and gauge equivalence classes of solutions to the $\mathcal{N}$-extended self-dual SYM equations to the level of the hierarchies, i.e., now we have a one-to-one correspondence between equivalence classes holomorphic vector bundles over the enhanced supertwistor space and gauge equivalence classes of solutions to the truncated $\mathcal{N}$-extended self-dual SYM hierarchy. Of course, by way of construction, the old correspondence is just a subset of this extension. In summary, we have the bijection

$$\mathcal{M}_{\rm hol}(\mathcal{P}^{3|\mathcal{N}}_{b,f}) \;\longleftrightarrow\; \mathcal{M}^{\mathcal{N}}_{\rm SDYMH}(b,f), \tag{4.17}$$

where $\mathcal{M}^{\mathcal{N}}_{\rm SDYMH}(b,f)$ denotes the moduli space of solutions to the respective truncated $\mathcal{N}$-extended self-dual SYM hierarchy.

4.3. Superfield equations of motion

So far, we have written down the truncated self-dual SYM hierarchies (4.13) quite abstractly as the compatibility conditions of the linear system (4.11). By recalling the discussion of the $\mathcal{N}$-extended super SDYM equations presented at the end of subsection 2.1, the next step in our discussion is to look for the equations of motion on superfield level equivalent to (4.13). To do this, we need to identify the (super)field content. At first sight, we expect to find – as *fundamental* field content (in a covariant formulation) – the field content of the $\mathcal{N}$-extended self-dual SYM theory plus a tower of additional fields which depends on the parameters b_α and f_i. However, as we shortly realize, this will not be entirely true. Instead we shall find that for $f_i > \frac{1}{2}$ certain combinations of the $\mathcal{A}^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f_i-1}}$ play the role of potentials for a lot of the naively expected fields, such that those combinations should be regarded as fundamental fields.

For the rest of this section, we shall for simplicity consider the case where $b_1 = b_2 \equiv b$ and $f_1 = \cdots = f_{\mathcal{N}} \equiv f$. For a moment, let us also introduce a shorthand index notation

$$\mathcal{A}_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2b-1}} \;\equiv\; \mathcal{A}_{\alpha\dot\alpha\dot A} \qquad\text{and}\qquad \mathcal{A}^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f-1}} \;\equiv\; \mathcal{A}^I_{\dot\alpha}, \tag{4.18}$$

which simplifies the subsequent formulas.

First, we point out that the equations (4.13) can concisely be rewritten as

$$\begin{gathered}[\nabla_{\alpha\dot\alpha\dot A}, \nabla_{\beta\dot\beta\dot B}] + [\nabla_{\alpha\dot\beta\dot A}, \nabla_{\beta\dot\alpha\dot B}] \;=\; 0, \qquad [\nabla^I_{\dot\alpha}, \nabla_{\beta\dot\beta\dot B}] + [\nabla^I_{\dot\beta}, \nabla_{\beta\dot\alpha\dot B}] \;=\; 0,\\ \{\nabla^I_{\dot\alpha}, \nabla^J_{\dot\beta}\} + \{\nabla^I_{\dot\beta}, \nabla^J_{\dot\alpha}\} \;=\; 0,\end{gathered} \tag{4.19}$$

which translates to the following superfield definitions

$$[\nabla_{\alpha\dot{\alpha}\dot{A}}, \nabla_{\beta\dot{\beta}\dot{B}}] \equiv \epsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\dot{A}\beta\dot{B}}, \tag{4.20a}$$

$$[\nabla^I_{\dot{\alpha}}, \nabla_{\beta\dot{\beta}\dot{B}}] \equiv \epsilon_{\dot{\alpha}\dot{\beta}} \chi^I_{\beta\dot{B}}, \tag{4.20b}$$

$$\{\nabla^I_{\dot{\alpha}}, \nabla^J_{\dot{\beta}}\} \equiv 2\epsilon_{\dot{\alpha}\dot{\beta}} W^{IJ}. \tag{4.20c}$$

Note that quite generally we have

$$\begin{aligned} \mathcal{F}_{\alpha\dot{\alpha}\dot{A}\beta\dot{\beta}\dot{B}} &= [\nabla_{\alpha\dot{\alpha}\dot{A}}, \nabla_{\beta\dot{\beta}\dot{B}}] = \tfrac{1}{2}(\mathcal{F}_{\alpha\dot{\alpha}\dot{A}\beta\dot{\beta}\dot{B}} - \mathcal{F}_{\beta\dot{\beta}\dot{B}\alpha\dot{\alpha}\dot{A}}) \\ &= \epsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\dot{A}\beta\dot{B}} + \epsilon_{\alpha\beta} f_{\dot{\alpha}\dot{A}\dot{\beta}\dot{B}} + \mathcal{F}_{\alpha\dot{\alpha}[\dot{A}\beta\dot{\beta}\dot{B}]}, \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} f_{\alpha\dot{A}\beta\dot{B}} &\equiv -\tfrac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}} \mathcal{F}_{\alpha\dot{\alpha}\dot{A}\beta\dot{\beta}\dot{B}}, \\ f_{\dot{\alpha}\dot{A}\dot{\beta}\dot{B}} &\equiv \tfrac{1}{2}\epsilon^{\alpha\beta} \mathcal{F}_{\alpha\dot{\alpha}\dot{A}\beta\dot{\beta}\dot{B}}, \\ \mathcal{F}_{\alpha\dot{\alpha}[\dot{A}\beta\dot{\beta}\dot{B}]} &\equiv \tfrac{1}{2}(\mathcal{F}_{\alpha\dot{\alpha}\dot{A}\beta\dot{\beta}\dot{B}} - \mathcal{F}_{\alpha\dot{\alpha}\dot{B}\beta\dot{\beta}\dot{A}}). \end{aligned} \tag{4.22}$$

Equation (4.21) can be simplified further to

$$\mathcal{F}_{\alpha\dot{\alpha}\dot{A}\beta\dot{\beta}\dot{B}} = \epsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\dot{A}\beta\dot{B}} + \epsilon_{\alpha\beta} f_{\dot{\alpha}(\dot{A}\dot{\beta}\dot{B})} + \mathcal{F}_{(\alpha\dot{\alpha}[\dot{A}\beta)\dot{\beta}\dot{B}]}. \tag{4.23}$$

Therefore, equation (4.20a) implies that

$$f_{\dot{\alpha}(\dot{A}\dot{\beta}\dot{B})} = 0 \qquad \text{and} \qquad \mathcal{F}_{(\alpha\dot{\alpha}[\dot{A}\beta)\dot{\beta}\dot{B}]} = 0, \tag{4.24}$$

which are the first two of the superfield equations of motion. We point out that for the choice $\dot{A} = \dot{B} = (\dot{1}\cdots\dot{1})$ the set (4.24) represents nothing but the ordinary SDYM equations (cf. the first equation of (2.17)).

Next, we consider the Bianchi identity for the triple $(\nabla_{\alpha\dot{\alpha}\dot{A}}, \nabla^I_{\dot{\beta}}, \nabla_{\gamma\dot{\gamma}\dot{C}})$. We find

$$\nabla^I_{\dot{\alpha}} f_{\alpha\dot{A}\beta\dot{B}} = \nabla_{\alpha\dot{\alpha}\dot{A}} \chi^I_{\beta\dot{B}}. \tag{4.25}$$

From this equation we deduce another two field equations, namely

$$\epsilon^{\alpha\beta} \nabla_{\alpha\dot{\alpha}(\dot{A}} \chi^I_{\beta\dot{B})} = 0 \qquad \text{and} \qquad \nabla_{(\alpha\dot{\alpha}[\dot{A}} \chi^I_{\beta)\dot{B}]} = 0. \tag{4.26}$$

Now the Bianchi identity for $(\nabla_{\alpha\dot{\alpha}\dot{A}}, \nabla^I_{\dot{\beta}}, \nabla^J_{\dot{\gamma}})$ says that

$$\nabla_{\alpha\dot{\alpha}\dot{A}} W^{IJ} = \tfrac{1}{2}\nabla^I_{\dot{\alpha}} \chi^J_{\alpha\dot{A}}. \tag{4.27}$$

Applying $\nabla_{\beta\dot{\beta}\dot{B}}$ to (4.27), we obtain upon (anti)symmetrization the following two equations of motion

$$\begin{aligned} \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}(\dot{A}}\nabla_{\beta\dot{\beta}\dot{B})}W^{IJ} + \epsilon^{\alpha\beta}\{\chi^I_{\alpha(\dot{A}}, \chi^J_{\beta\dot{B})}\} &= 0, \\ \epsilon^{\dot{\alpha}\dot{\beta}}\nabla_{(\alpha\dot{\alpha}[\dot{A}}\nabla_{\beta)\dot{\beta}\dot{B}]}W^{IJ} + \{\chi^I_{(\alpha[\dot{A}}, \chi^J_{\beta)\dot{B}]}\} &= 0. \end{aligned} \tag{4.28}$$

The Bianchi identity for the combination $(\nabla^I_{\dot{\alpha}}, \nabla^J_{\dot{\beta}}, \nabla^K_{\dot{\gamma}})$ shows that $\nabla^I_{\dot{\alpha}}W^{JK}$ determines a superfield which is totally antisymmetric in the indices IJK, i.e.,

$$\nabla^I_{\dot{\alpha}}W^{JK} \equiv \chi^{IJK}_{\dot{\alpha}}. \tag{4.29}$$

Upon acting on both sides by $\nabla_{\alpha\dot{\alpha}\dot{A}}$ and contracting the dotted indices, we obtain the field equation for $\chi^{IJK}_{\dot{\alpha}}$

$$\epsilon^{\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}\dot{A}}\chi^{IJK}_{\dot{\beta}} - 3[W^{[IJ}, \chi^{K]}_{\alpha\dot{A}}] = 0. \tag{4.30}$$

The application of $\nabla^I_{\dot{\alpha}}$ to $\chi^{JKL}_{\dot{\beta}}$ and symmetrization in $\dot{\alpha}$ and $\dot{\beta}$ leads by virtue of (4.29) to a new superfield which is totally antisymmetric in $IJKL$,

$$\nabla^I_{(\dot{\alpha}}\chi^{JKL}_{\dot{\beta})} \equiv G^{IJKL}_{\dot{\alpha}\dot{\beta}}. \tag{4.31}$$

Furthermore, some algebraic manipulations show that

$$\nabla^I_{\dot{\alpha}}\chi^{JKL}_{\dot{\beta}} = \nabla^I_{(\dot{\alpha}}\chi^{JKL}_{\dot{\beta})} + \nabla^I_{[\dot{\alpha}}\chi^{JKL}_{\dot{\beta}]} = G^{IJKL}_{\dot{\alpha}\dot{\beta}} + 3\epsilon_{\dot{\alpha}\dot{\beta}}[W^{I[J}, W^{KL]}], \tag{4.32}$$

where equation (4.29) and the definition (4.31) have been used. From this equation, the equation of motion for the superfield $G^{IJKL}_{\dot{\alpha}\dot{\beta}}$ can readily be derived. We obtain

$$\epsilon^{\dot{\alpha}\dot{\gamma}}\nabla_{\alpha\dot{\alpha}\dot{A}}G^{IJKL}_{\dot{\beta}\dot{\gamma}} + 4\{\chi^{[I}_{\alpha\dot{A}}, \chi^{JKL]}_{\dot{\beta}}\} + 6[W^{[JK}, \nabla_{\alpha\dot{\beta}\dot{A}}W^{LI]}] = 0. \tag{4.33}$$

As (4.29) implies the existence of the superfield $G^{IJKL}_{\dot{\alpha}\dot{\beta}}$, the definition (4.31) determines a new superfield $\psi^{IJKLM}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}$ being totally antisymmetric in $IJKLM$ and totally symmetric in $\dot{\alpha}\dot{\beta}\dot{\gamma}$, i.e.,

$$\nabla^I_{(\dot{\alpha}}G^{JKLM}_{\dot{\beta}\dot{\gamma})} \equiv \psi^{IJKLM}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}. \tag{4.34}$$

It is easily shown that

$$\nabla^I_{\dot{\alpha}}G^{JKLM}_{\dot{\beta}\dot{\gamma}} = \nabla^I_{(\dot{\alpha}}G^{JKLM}_{\dot{\beta}\dot{\gamma})} - \tfrac{2}{3}\epsilon_{\dot{\alpha}(\dot{\beta}}\epsilon^{\dot{\delta}\dot{\epsilon}}\nabla^I_{\dot{\delta}}G^{JKLM}_{\dot{\epsilon}\dot{\gamma})}. \tag{4.35}$$

After some tedious algebra, we obtain from (4.35) the formula

$$\nabla^I_{\dot{\alpha}}G^{JKLM}_{\dot{\beta}\dot{\gamma}} = \psi^{IJKLM}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} + \tfrac{4}{3}\epsilon_{\dot{\alpha}(\dot{\beta}}\left(4[W^{I[J}, \chi^{KLM]}_{\dot{\gamma})}] + 3[\chi^{I[JK}_{\dot{\gamma})}, W^{LM]}]\right), \tag{4.36}$$

where the definition (4.34) has been substituted. This equation in turn implies the equation of motion for $\psi^{IJKLM}_{\dot\alpha\dot\beta\dot\gamma}$,

$$\epsilon^{\dot\alpha\dot\delta}\nabla_{\alpha\dot\alpha\dot A}\psi^{IJKLM}_{\dot\beta\dot\gamma\dot\delta} + 5[\chi^{[I}_{\alpha\dot A}, G^{JKLM]}_{\dot\beta\dot\gamma}] + \\ + \tfrac{40}{3}[\nabla_{\alpha(\dot\beta\dot A}W^{[IJ}, \chi^{KLM]}_{\dot\gamma)}] - \tfrac{20}{3}[W^{[IJ}, \nabla_{\alpha(\dot\beta\dot A}\chi^{KLM]}_{\dot\gamma)}] \;=\; 0, \tag{4.37}$$

which follows after a somewhat lengthy calculation.

Now one can continue this procedure of defining superfields via the action of $\nabla^I_{\dot\alpha}$ and of finding the corresponding equations of motion. Generically, the number of fields one obtains in this way is determined by the parameter f, i.e., the most one can get is

$$\psi^{I_1\cdots I_{2\mathcal{N}f}}_{\dot\alpha_1\cdots\dot\alpha_{2\mathcal{N}f-2}},$$

which is, as before, totally antisymmetric in $I_1\cdots I_{2\mathcal{N}f}$ and totally symmetric in $\dot\alpha_1\cdots\dot\alpha_{2\mathcal{N}f-2}$.

Let us collect the superfield equations of motion for the $\mathcal{N}$-extended self-dual SYM hierarchy:

$$\begin{aligned}
f_{\dot\alpha(\dot A\dot\beta\dot B)} \;=\; 0 \qquad \text{and} \qquad \mathcal{F}_{(\alpha\dot\alpha[\dot A\beta)\dot\beta\dot B]} &\;=\; 0,\\
\epsilon^{\alpha\beta}\nabla_{\alpha\dot\alpha(\dot A}\chi^I_{\beta\dot B)} \;=\; 0 \qquad \text{and} \qquad \nabla_{(\alpha\dot\alpha[\dot A}\chi^I_{\beta)\dot B]} &\;=\; 0,\\
\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha(\dot A}\nabla_{\beta\dot\beta\dot B)}W^{IJ} + \epsilon^{\alpha\beta}\{\chi^I_{\alpha(\dot A}, \chi^J_{\beta\dot B)}\} &\;=\; 0,\\
\epsilon^{\dot\alpha\dot\beta}\nabla_{(\alpha\dot\alpha[\dot A}\nabla_{\beta)\dot\beta\dot B]}W^{IJ} + \{\chi^I_{(\alpha[\dot A}, \chi^J_{\beta)\dot B]}\} &\;=\; 0,\\
\epsilon^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha\dot A}\chi^{IJK}_{\dot\beta} - 3[W^{[IJ}, \chi^{K]}_{\alpha\dot A}] &\;=\; 0,\\
\epsilon^{\dot\alpha\dot\gamma}\nabla_{\alpha\dot\alpha\dot A}G^{IJKL}_{\dot\beta\dot\gamma} + 4\{\chi^{[I}_{\alpha\dot A}, \chi^{JKL]}_{\dot\beta}\} + 6[W^{[JK}, \nabla_{\alpha\dot\beta\dot A}W^{LI]}] &\;=\; 0,\\
&\;\;\vdots\\
\epsilon^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha\dot A}\psi^{I_1\cdots I_{2\mathcal{N}f}}_{\dot\beta\dot\alpha_1\cdots\dot\alpha_{2\mathcal{N}f-3}} + J^{I_1\cdots I_{2\mathcal{N}f}}_{\alpha\dot A\dot\alpha_1\cdots\dot\alpha_{2\mathcal{N}f-3}} &\;=\; 0,
\end{aligned} \tag{4.38}$$

where the currents $J^{I_1\cdots I_{2\mathcal{N}f}}_{\alpha\dot A\dot\alpha_1\cdots\dot\alpha_{2\mathcal{N}f-3}}$ are determined in an obvious manner. Clearly, the system (4.38) contains as a subset the $\mathcal{N}$-extended self-dual SYM equations. In particular, for the choice $b = f = \frac{1}{2}$ it reduces to (2.17). Altogether, we have obtained the field content of the $\mathcal{N}$-extended self-dual SYM theory plus a number of additional fields together with their superfield equations of motion.

However, as we have already indicated, this is not the end of the story. The system (4.38), though describing the truncated hierarchy, contains a lot of redundant information.

So, it should not be regarded as the *fundamental* system displaying the truncated hierarchy. Namely, the use of the shorthand index notation (4.18) does not entirely reflect all of the possible index symmetry properties of the appearing superfields. In order to incorporate all possibilities, we instead need to write out the explicit form of $I, J, K, \ldots$.

As before, let us impose the transversal gauge condition

$$\eta_i^{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f-1}}\mathcal{A}^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f-1}} \;=\; 0, \tag{4.39}$$

which again reduces super gauge transformations to ordinary ones. Note that in (4.39) only $\mathcal{A}^i_{(\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f-1})}$ contributes, since the fermionic coordinates are totally symmetric under an exchange of their dotted indices. The condition (4.39) then allows to define the recursion operator $\mathcal{D}$ according to

$$\mathcal{D} \;\equiv\; \eta_i^{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f-1}}\nabla^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f-1}} \;=\; \eta_i^{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f-1}}\partial^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f-1}}, \tag{4.40}$$

i.e., $\mathcal{D}$ is a positive definite homogeneity operator. Equation (4.20c) yields

$$\begin{aligned}(1+\mathcal{D})\mathcal{A}^i_{(\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f-1})} \;&=\; -2\epsilon_{\dot\beta(\dot\alpha}\eta_j^{\dot\beta\dot\beta_1\cdots\dot\beta_{2f-1}}W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{2f-1})\,\dot\beta_1\cdots\dot\beta_{2f-1}},\\ \mathcal{D}\mathcal{A}^i_{[\dot\alpha,\dot\beta]\dot\alpha_1\cdots\dot\alpha_{2f-2}} \;&=\; \epsilon_{\dot\alpha\dot\beta}\eta_j^{\dot\gamma\dot\beta_1\cdots\dot\beta_{2f-1}}W^{ij}_{\dot\gamma\dot\alpha_1\cdots\dot\alpha_{2f-2}\,\dot\beta_1\cdots\dot\beta_{2f-1}},\end{aligned} \tag{4.41}$$

which states that $\mathcal{A}^i_{(\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f-1})}$ does not have a zeroth order component in the η-expansion while $\mathcal{A}^i_{[\dot\alpha,\dot\beta]\dot\alpha_1\cdots\dot\alpha_{2f-2}}$ does. Therefore, we obtain as a fundamental superfield

$$\phi^i_{\dot\alpha_1\cdots\dot\alpha_{2f-2}} \;\equiv\; \mathcal{A}^i_{[\dot 1,\dot 2]\dot\alpha_1\cdots\dot\alpha_{2f-2}}, \qquad \text{for} \qquad f>\tfrac{1}{2}. \tag{4.42}$$

Note that as $\phi^i_{\dot\alpha_1\cdots\dot\alpha_{2f-2}}=\phi^i_{(\dot\alpha_1\cdots\dot\alpha_{2f-2})}$ it defines for each i a spin $f-1$ superfield of odd Graßmann parity.

Equation (4.20b) reads explicitly as

$$[\nabla^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f-1}},\nabla_{\beta\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1}}] \;=\; \epsilon_{\dot\alpha\dot\beta}\chi^i_{\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\beta\,\dot\beta_1\cdots\dot\beta_{2b-1}}.$$

The contraction with $\epsilon^{\dot\alpha\dot\alpha_1}$ shows that

$$\chi^i_{\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\beta\,\dot\beta_1\cdots\dot\beta_{2b-1}} \;=\; 2\nabla_{\beta(\dot\alpha_1\dot\beta_1\cdots\dot\beta_{2b-1}}\phi^i_{\dot\alpha_2\cdots\dot\alpha_{2f-1})}, \tag{4.43}$$

where the symmetrization is only ment between the $\dot\alpha_1\cdots\dot\alpha_{2f-1}$. Therefore, the superfield $\chi^i_{\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\beta\,\dot\beta_1\cdots\dot\beta_{2b-1}}$ cannot be regarded as a fundamental field – the superfield (4.42) plays the role of a potential for the former.

Next, we discuss the superfield $W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}}$. To show which combinations of it are really fundamental, we need some preliminaries. Consider the index set

$$\dot\alpha_1\cdots\dot\alpha_n\dot\beta_1\cdots\dot\beta_n,$$

which is separately totally symmetric in $\dot\alpha_1\cdots\dot\alpha_n$ and $\dot\beta_1\cdots\dot\beta_n$, respectively. Then we have the useful formula

$$\begin{aligned}\dot\alpha_1\cdots\dot\alpha_n\dot\beta_1\cdots\dot\beta_n \;&=\; (\dot\alpha_1\cdots\dot\alpha_n\dot\beta_1\cdots\dot\beta_n) \;+\; \sum \text{ all possible contractions}\\ &=\; (\dot\alpha_1\cdots\dot\alpha_n\dot\beta_1\cdots\dot\beta_n) \;+\\ &\qquad\qquad +\; A_1\sum_{i,j}\dot\alpha_1\cdots\overbrace{\dot\alpha_i\cdots\dot\alpha_n\dot\beta_1\cdots\dot\beta_j}\cdots\dot\beta_n \;+\;\cdots,\end{aligned} \tag{4.44}$$

where the parantheses denote, as before, symmetrization of the enclosed indices and "contraction" means antisymmetrization in the respective index pair. The A_i for $i=1,\ldots,n$ are combinatorial coefficients, whose explicit form is not needed in the sequel. The proof of (4.44) is quite similar to the one of the Wick theorem and we thus leave it to the interested reader. Equation (4.20c) is explicitly given by

$$\{\nabla^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f-1}},\nabla^j_{\dot\beta\dot\beta_1\cdots\dot\beta_{2f-1}}\} \;=\; 2\epsilon_{\dot\alpha\dot\beta}W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}}.$$

After contraction with $\epsilon^{\dot\alpha\dot\alpha_1}$ we obtain

$$W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}} \;=\; -\nabla^j_{\dot\alpha_1\dot\beta_1\cdots\dot\beta_{2f-1}}\phi^i_{\dot\alpha_2\cdots\dot\alpha_{2f-1}}, \tag{4.45}$$

where the definition (4.42) has been inserted. Contracting this equation with $\epsilon^{\dot\alpha_1\dot\beta_1}$, we get

$$\epsilon^{\dot\alpha_1\dot\beta_1}W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}} \;=\; -2\{\phi^i_{\dot\alpha_2\cdots\dot\alpha_{2f-1}},\phi^j_{\dot\beta_2\cdots\dot\beta_{2f-1}}\}. \tag{4.46}$$

Thus, we conclude that $W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{2f-2}[\dot\alpha_{2f-1}\dot\beta_1]\dot\beta_2\cdots\dot\beta_{2f-1}}$ is a composite field and hence not a fundamental one. Using formula (4.44), we may schematically write

$$W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}} \;=\; W^{ij}_{(\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1})}+\sum \text{ all possible contractions.} \tag{4.47}$$

The contraction terms in (4.47), however, are all composite expressions due to (4.46). Therefore, only the superfield

$$W^{ij}_{(\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1})} \;=\; W^{[ij]}_{(\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1})} \tag{4.48}$$

is fundamental. For each combination $[ij]$ it represents a Graßmann even superfield with spin $2f-1$.

Now we need to consider the superfield defined in (4.29),

$$\chi^{ijk}_{\dot\alpha\,\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}\,\dot\gamma_1\cdots\dot\gamma_{2f-1}} \;=\; \nabla^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2f-1}} W^{ij}_{\dot\beta_1\cdots\dot\beta_{2f-1}\,\dot\gamma_1\cdots\dot\gamma_{2f-1}}.$$

By extending the formula (4.44) to the index triple

$$\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}\,\dot\gamma_1\cdots\dot\gamma_{2f-1}$$

and by utilizing the symmetry properties of $\chi^{ijk}_{\dot\alpha\,\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}\,\dot\gamma_1\cdots\dot\gamma_{2f-1}}$, one can show, by virtue of the above arguments, that only the combination

$$\chi^{ijk}_{(\dot\alpha\,\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}\,\dot\gamma_1\cdots\dot\gamma_{2f-1})} \;=\; \chi^{[ijk]}_{(\dot\alpha\,\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}\,\dot\gamma_1\cdots\dot\gamma_{2f-1})} \tag{4.49}$$

of $\chi^{ijk}_{\dot\alpha\,\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}\,\dot\gamma_1\cdots\dot\gamma_{2f-1}}$ remains as a fundamental superfield. It defines for each $[ijk]$ a Graßmann odd superfield with spin $3f-1$.

Repeating this procedure, we deduce from the definition (4.31) that

$$\begin{aligned} &G^{ijkl}_{(\dot\alpha\dot\beta\,\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}\,\dot\gamma_1\cdots\dot\gamma_{2f-1}\,\dot\delta_1\cdots\dot\delta_{2f-1})} \\ &\qquad\qquad = \; G^{[ijkl]}_{(\dot\alpha\dot\beta\,\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}\,\dot\gamma_1\cdots\dot\gamma_{2f-1}\,\dot\delta_1\cdots\dot\delta_{2f-1})} \end{aligned} \tag{4.50}$$

is fundamental and it represents one[35] spin $4f-1$ superfield which is Graßmann even. All higher order fields, such as (4.34), yield no further fundamental fields due to the antisymmetrization of $ijklm$, etc. In summary, the fundamental field content of the truncated self-dual SYM hierarchies is given by

$$\begin{aligned} &\mathcal{A}_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2b-1}}, \quad \phi^i_{\dot\alpha_1\cdots\dot\alpha_{2f-2}}, \quad W^{[ij]}_{(\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1})}, \\ &\chi^{[ijk]}_{(\dot\alpha\,\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}\,\dot\gamma_1\cdots\dot\gamma_{2f-1})}, \quad G^{[ijkl]}_{(\dot\alpha\dot\beta\,\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}\,\dot\gamma_1\cdots\dot\gamma_{2f-1}\,\dot\delta_1\cdots\dot\delta_{2f-1})}, \end{aligned} \tag{4.51}$$

where we assume that $f > \frac{1}{2}$.[36] All other naively expected fields, which for instance appear in (4.38), are composite expressions of the above fields.

It remains to find the superfield equations of motion for the fields (4.51). This, however, is easily done since we have already derived (4.38). By following the lines which led

[35] Note that we assume $\mathcal{N} \leq 4$.

[36] For $f = \frac{1}{2}$, the field $\phi^i_{\dot\alpha_1\cdots\dot\alpha_{2f-2}}$ must be replaced by χ^i_α.

to (4.38) and by taking into account the definition (4.42), the system (4.38) reduces for $f > \frac{1}{2}$ to

$$\begin{aligned}
f_{\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{2b-1}\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1})} \;=\; 0 \qquad \text{and} \qquad \mathcal{F}_{(\alpha\dot\alpha[\dot\alpha_1\cdots\dot\alpha_{2b-1}\beta)\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1}]} &= 0,\\
\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{2b-1}}\nabla_{\beta\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1})}\phi^i_{\dot\gamma_1\cdots\dot\gamma_{2f-2}} &= 0,\\
\nabla_{(\alpha\dot\alpha[\dot\alpha_1\cdots\dot\alpha_{2b-1}}\nabla_{\beta)(\dot\gamma_1\dot\beta_1\cdots\dot\beta_{2b-1}]}\phi^i_{\dot\gamma_2\cdots\dot\gamma_{2f-1})} &= 0,\\
\epsilon^{\dot\alpha\dot\beta_1}\nabla_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2b-1}}W^{[ij]}_{(\dot\beta_1\cdots\dot\beta_{4f-2})} - \qquad&\\
-\, 2\{\phi^{[i}_{(\dot\beta_2\cdots\dot\beta_{2f-1}},\nabla_{\alpha\dot\beta_{2f}\dot\alpha_1\cdots\dot\alpha_{2b-1}}\phi^{j]}_{\dot\beta_{2f+1}\cdots\dot\gamma_{4f-2})}\} &= 0,\\
\epsilon^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\beta_{2b-1}}\chi^{[ijk]}_{(\dot\beta\dot\beta_1\cdots\dot\beta_{6f-3})} - \qquad&\\
-\, 6[W^{[ij}_{(\dot\beta_1\cdots\dot\beta_{4f-2}},\nabla_{\alpha\dot\beta_{4f-1}\dot\alpha_1\cdots\dot\alpha_{2b-1}}\phi^{k]}_{\dot\beta_{4f}\cdots\dot\beta_{6f-3})}] &= 0,\\
\epsilon^{\dot\alpha\dot\gamma}\nabla_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2b-1}}G^{[ijkl]}_{(\dot\beta\dot\gamma\dot\beta_1\cdots\dot\beta_{8f-4})} + \qquad&\\
-\, 8\{\nabla_{\alpha(\dot\beta_1\dot\alpha_1\cdots\dot\alpha_{2b-1}}\phi^{[i}_{\dot\beta_2\cdots\dot\beta_{2f-1}},\chi^{ijk]}_{\dot\beta\dot\beta_{2f}\cdots\dot\beta_{8f-4})}\} - \qquad&\\
-\, 6[W^{[ij}_{(\dot\beta_1\cdots\dot\beta_{4f-2}},\nabla_{\alpha\dot\beta\dot\alpha_1\cdots\dot\alpha_{2b-1}}W^{kl]}_{\dot\beta_{4f-1}\cdots\dot\beta_{8f-4})}] &= 0,
\end{aligned} \tag{4.52}$$

which are the superfield equations of motion for the truncated $\mathcal{N}$-extended self-dual SYM hierarchy.

4.4. Equivalence of the field equations and the compatibility conditions

Above we have derived the superfield equations of motion. What remains is to show how the superfields (4.51) are expressed in terms of their zeroth order components

$$\overset{\circ}{\mathcal{A}}_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2b-1}},\quad \overset{\circ}{\phi}{}^i_{\dot\alpha_1\cdots\dot\alpha_{2f-2}},\quad \overset{\circ}{W}{}^{[ij]}_{(\dot\alpha_1\cdots\dot\alpha_{4f-2})},\quad \overset{\circ}{\chi}{}^{[ijk]}_{(\dot\alpha_1\cdots\dot\alpha_{6f-2})},\quad \overset{\circ}{G}{}^{[ijkl]}_{(\dot\alpha_1\cdots\dot\alpha_{8f-2})}, \tag{4.53}$$

and furthermore that the field equations on $\mathbb{C}^4$ (or $\mathbb{R}^4$ after reality conditions have been imposed)[37], i.e., those equations that are obtained from the set (4.52) by projecting onto the zeroth order components (4.53) of the superfields (4.51), imply the compatibility conditions (4.19). We will, however, be not too explicit in showing this equivalence, since the argumentation goes along similar lines as those given for the $\mathcal{N}$-extended self-dual SYM equations (cf. [54]). Here, we just sketch the idea.

[37] Note that in total we have $4b+2$ complex bosonic coordinates. Here, we interpret the spacetime $\mathbb{C}^4$ as a subset of $\mathbb{C}^{4b+2} \cong \mathbb{C}^4 \oplus \mathbb{C}^{4b-2}$ (see also the discussion in subsection 4.1), i.e., the remaining $4b-2$ bosonic coordinates are regarded as additional moduli of the fields on $\mathbb{C}^4$.

In order to write down the superfield expansions, remember that we have imposed the gauge (4.39) which led to the recursion operator $\mathcal{D}$ according to (4.40). Using the formulas (4.20), (4.27), (4.29), (4.31), (4.34) and (4.43), we obtain the following recursion relations:

$$\begin{aligned}
(1+\mathcal{D})\mathcal{A}^i_{(\dot\alpha_1\cdots\dot\alpha_{2f})} &= -2\epsilon_{\dot\beta(\dot\alpha_1}\eta_j^{\dot\beta\dot\beta_1\cdots\dot\beta_{2f-1}}W^{ij}_{\dot\alpha_2\cdots\dot\alpha_{2f})\,\dot\beta_1\cdots\dot\beta_{2f-1}}, && (4.54a)\\
\mathcal{D}\mathcal{A}_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2b-1}} &= -2\epsilon_{\dot\alpha\dot\beta}\eta_j^{\dot\beta\dot\beta_1\cdots\dot\beta_{2f-1}}\nabla_{\alpha\dot\beta_1\dot\alpha_1\cdots\dot\alpha_{2b-1}}\phi^i_{\dot\beta_2\cdots\dot\beta_{2f-1}}, && (4.54b)\\
\mathcal{D}\phi^i_{\dot\alpha_1\cdots\dot\alpha_{2f-2}} &= \eta_j^{\dot\gamma\dot\beta_1\cdots\dot\beta_{2f-1}}W^{ij}_{\dot\gamma\dot\alpha_1\cdots\dot\alpha_{2f-2}\,\dot\beta_1\cdots\dot\beta_{2f-1}}, && (4.54c)\\
\mathcal{D}W^{[ij]}_{(\dot\alpha_1\cdots\dot\alpha_{4f-2})} &= \eta_k^{\dot\beta\dot\beta_1\cdots\dot\beta_{2f-1}}\chi^{[ij]k}_{\dot\beta(\dot\alpha_1\cdots\dot\alpha_{4f-2})\,\dot\beta_1\cdots\dot\beta_{2f-1}}, && (4.54d)\\
\mathcal{D}\chi^{[ijk]}_{(\dot\alpha_1\cdots\dot\alpha_{6f-2})} &= \eta_l^{\dot\beta\dot\beta_1\cdots\dot\beta_{2f-1}}G^{[ijk]l}_{\dot\beta(\dot\alpha_1\cdots\dot\alpha_{6f-2})\,\dot\beta_1\cdots\dot\beta_{2f-1}}+\\
&\quad+\ 3\epsilon_{\dot\beta(\dot\alpha_1}\eta_l^{\dot\beta\dot\beta_1\cdots\dot\beta_{2f-1}}[W^{l[i}_{\dot\beta_1\cdots\dot\beta_{2f-1}\,\dot\alpha_2\cdots\dot\alpha_{2f}},W^{jk]}_{\dot\alpha_{2f+1}\cdots\dot\alpha_{6f-2})}], && (4.54e)\\
\mathcal{D}G^{[ijkl]}_{(\dot\alpha_1\cdots\dot\alpha_{8f-2})} &= \eta_m^{\dot\beta\dot\beta_1\cdots\dot\beta_{2f-1}}\psi^{[ijkl]m}_{\dot\beta(\dot\alpha_1\cdots\dot\alpha_{8f-2})\,\dot\beta_1\cdots\dot\beta_{2f-1}}+\\
&\quad+\ \tfrac{4}{3}\epsilon_{\dot\beta(\dot\alpha_1}\eta_m^{\dot\beta\dot\beta_1\cdots\dot\beta_{2f-1}}\Big(4[W^{m[i}_{\dot\beta_1\cdots\dot\beta_{2f-1}\,\dot\alpha_2\cdots\dot\alpha_{2f}},\chi^{jkl]}_{\dot\alpha_{2f+1}\cdots\dot\alpha_{8f-2})}]+\\
&\qquad+3[\chi^{m[ij}_{\dot\alpha_2\,\dot\beta_1\cdots\dot\beta_{2f-1}\,\dot\alpha_3\cdots\dot\alpha_{4f-4}},W^{kl]}_{\dot\alpha_{4f-4}\cdots\dot\alpha_{8f-2})}]\Big)\,. && (4.54f)
\end{aligned}$$

An explanation of these formulas is in order. The right hand sides of the equations (4.54) depend not only on the fundamental fields (4.51) but also on composite expressions of those fields: For instance, consider the recursion relation (4.54c) of the field $\phi^i_{\dot\alpha_1\cdots\dot\alpha_{2f-2}}$. The right hand side of (4.54c) depends on the superfield $W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1}}$. However, as we learned in (4.47), it can be rewritten as the fundamental field $W^{[ij]}_{(\dot\alpha_1\cdots\dot\alpha_{2f-1}\,\dot\beta_1\cdots\dot\beta_{2f-1})}$ plus contraction terms which are of the form (4.46). Similar arguments hold for the other recursion relations. Therefore, the right hand sides of (4.54) can solely be written in terms of the fundamental fields. However, as these formulas in terms of the fundamental fields will look pretty messy, we refrain from writing them down but always have in mind their explicit expansions. Note that in (4.54f) the field $\psi^{[ijkl]m}_{\dot\beta(\dot\alpha_1\cdots\dot\alpha_{8f-2})\,\dot\beta_1\cdots\dot\beta_{2f-1}}$ consists only of composite expressions of the fields (4.46). Using the equations (4.54), one can now straightforwardly determine the superfield expansions by a successive application of the recursion operator $\mathcal{D}$, since if one knows the expansions to n-th order in the fermionic coordinates, the recursions (4.54) yield them to $(n+1)$-th order, because of the positivity of $\mathcal{D}$. But again, this procedure will lead to both unenlightning and complicated looking expressions, so we do not present them here.

Finally, the recursion operator can be used to show the equivalence between the field equations and the constraint equations (2.13). This can be done inductively, i.e., one first assumes that the equations (4.52) hold to n-th order in the fermionic coordinates, then one applies $k+\mathcal{D}$ to (4.52), where $k \in \mathbb{N}_0$ is some properly chosen integer, and shows that they also hold to $(n+1)$-th order. To give an easy example, consider the curvature equations

$$f_{\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{2b-1}\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1})} \;=\; 0 \qquad \text{and} \qquad \mathcal{F}_{(\alpha\dot\alpha[\dot\alpha_1\cdots\dot\alpha_{2b-1}\beta)\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1}]} \;=\; 0 \tag{4.55}$$

and assume that they hold to n-th order. For the first equation one obtains

$$\begin{aligned}
\mathcal{D}f_{\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{2b-1}\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1})} \;&=\\
&= \tfrac{1}{2}\epsilon^{\alpha\beta}\mathcal{D}[\nabla_{\alpha\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{2b-1}}, \nabla_{\beta\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1})}]\\
&= \tfrac{1}{2}\epsilon^{\alpha\beta}\left(\nabla_{\alpha\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{2b-1}}\mathcal{D}\mathcal{A}_{\beta\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1})} - \nabla_{\beta\dot\beta(\dot\beta_1\cdots\dot\beta_{2b-1}}\mathcal{D}\mathcal{A}_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2b-1})}\right)\\
&= -\epsilon^{\alpha\beta}\Big(\epsilon_{\dot\beta\dot\gamma}\eta_i^{\dot\gamma\dot\gamma_1\cdots\dot\gamma_{2f-1}}\nabla_{\alpha\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{2b-1}}\nabla_{\beta\dot\gamma_1\dot\beta_1\cdots\dot\beta_{2b-1})}\phi^i_{\dot\gamma_2\cdots\dot\gamma_{2f-1}} -\\
&\qquad\qquad - \epsilon_{\dot\alpha\dot\gamma}\eta_i^{\dot\gamma\dot\gamma_1\cdots\dot\gamma_{2f-1}}\nabla_{\beta\dot\beta(\dot\beta_1\cdots\dot\beta_{2b-1}}\nabla_{\alpha\dot\gamma_1\dot\alpha_1\cdots\dot\alpha_{2b-1})}\phi^i_{\dot\gamma_2\cdots\dot\gamma_{2f-1}}\Big)\\
&= 0,
\end{aligned} \tag{4.56a}$$

and for the second one

$$\begin{aligned}
\mathcal{D}\mathcal{F}_{(\alpha\dot\alpha[\dot\alpha_1\cdots\dot\alpha_{2b-1}\beta)\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1}]} \;&=\\
&= \mathcal{D}[\nabla_{(\alpha\dot\alpha[\dot\alpha_1\cdots\dot\alpha_{2b-1}}, \nabla_{\beta)\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1}]}]\\
&= \nabla_{(\alpha\dot\alpha[\dot\alpha_1\cdots\dot\alpha_{2b-1}}\mathcal{D}\mathcal{A}_{\beta)\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1}]} - \nabla_{(\beta\dot\beta[\dot\beta_1\cdots\dot\beta_{2b-1}}\mathcal{D}\mathcal{A}_{\alpha)\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2b-1}]}\\
&= -2\epsilon_{\dot\beta\dot\gamma}\eta_i^{\dot\gamma\dot\gamma_1\cdots\dot\gamma_{2f-1}}\nabla_{(\alpha\dot\alpha[\dot\alpha_1\cdots\dot\alpha_{2b-1}}\nabla_{\beta)\dot\gamma_1\dot\beta_1\cdots\dot\beta_{2b-1}]}\phi^i_{\dot\gamma_2\cdots\dot\gamma_{2f-1}} +\\
&\qquad + 2\epsilon_{\dot\alpha\dot\gamma}\eta_i^{\dot\gamma\dot\gamma_1\cdots\dot\gamma_{2f-1}}\nabla_{(\beta\dot\beta[\dot\beta_1\cdots\dot\beta_{2b-1}}\nabla_{\alpha)\dot\gamma_1\dot\alpha_1\cdots\dot\alpha_{2b-1}]}\phi^i_{\dot\gamma_2\cdots\dot\gamma_{2f-1}}\\
&= 0,
\end{aligned} \tag{4.56b}$$

where we have used (4.54b) together with the superfield equations of $\phi^i_{\dot\alpha_1\cdots\dot\alpha_{2f-2}}$ given in (4.52) to n-th order, which shows that the equations (4.55) are indeed satisfied to $(n+1)$-th order. In the derivation (4.56a), we have used the identity

$$\begin{aligned}
\epsilon^{\alpha\beta}\nabla_{\alpha\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{2b-1}}\nabla_{\beta\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1})} \;&=\\
&= \tfrac{1}{2}\epsilon^{\alpha\beta}\left(\nabla_{\alpha\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{2b-1}}\nabla_{\beta\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1})} + \nabla_{\alpha\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{2b-1}}\nabla_{\beta\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1})}\right)\\
&= \tfrac{1}{2}\epsilon^{\alpha\beta}\left(-\nabla_{\alpha\dot\beta(\dot\alpha_1\cdots\dot\alpha_{2b-1}}\nabla_{\beta\dot\alpha\dot\beta_1\cdots\dot\beta_{2b-1})} + \nabla_{\alpha\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{2b-1}}\nabla_{\beta\dot\beta\dot\beta_1\cdots\dot\beta_{2b-1})}\right)\\
&= \tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\epsilon^{\dot\gamma\dot\delta}\nabla_{\alpha\dot\gamma(\dot\alpha_1\cdots\dot\alpha_{2b-1}}\nabla_{\beta\dot\delta\dot\beta_1\cdots\dot\beta_{2b-1})}
\end{aligned} \tag{4.57}$$

upon inserting the first equation of (4.55) (again to n-th order). Similar calculations apply for the remaining equations.

4.5. Leznov gauge (light cone formalism)

Again, we may introduce a Lie algebra valued potential Ψ leading to a simpler form of the truncated hierarchies (4.13). Moreover, the components of the gauge potential are then given by suitable derivatives of Ψ.

The gauge fixing condition in this case is as before

$$\psi_+ \;=\; 1 + \lambda_+ \Psi + \mathcal{O}(\lambda_+^2). \tag{4.58}$$

One readily verifies that this parametrization leads to

$$\begin{aligned} \mathcal{A}_{\alpha\dot{1}\dot{\alpha}_1\cdots\dot{\alpha}_{2b-1}} \;&=\; \partial_{\alpha\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{2b-1}}\Psi \qquad \text{and} \qquad \mathcal{A}_{\alpha\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{2b-1}} \;=\; 0, \\ \mathcal{A}^i_{\dot{1}\dot{\alpha}_1\cdots\dot{\alpha}_{2f-1}} \;&=\; \partial^i_{\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{2f-1}}\Psi \qquad \text{and} \qquad \mathcal{A}^i_{\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{2f-1}} \;=\; 0. \end{aligned} \tag{4.59}$$

Therefore, the truncated hierarchy (4.13) turns into the following system:

$$\begin{aligned} &\partial_{\alpha\dot{1}\dot{\alpha}_1\cdots\dot{\alpha}_{2b-1}}\partial_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{2b-1}}\Psi - \partial_{\beta\dot{1}\dot{\beta}_1\cdots\dot{\beta}_{2b-1}}\partial_{\alpha\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{2b-1}}\Psi + \\ &\qquad\qquad + \; [\partial_{\alpha\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{2b-1}}\Psi, \partial_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{2b-1}}\Psi] \;=\; 0, \\ &\partial^i_{\dot{1}\dot{\alpha}_1\cdots\dot{\alpha}_{2f-1}}\partial_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{2b-1}}\Psi - \partial_{\beta\dot{1}\dot{\beta}_1\cdots\dot{\beta}_{2b-1}}\partial^i_{\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{2f-1}}\Psi + \\ &\qquad\qquad + \; [\partial^i_{\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{2f-1}}\Psi, \partial_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{2b-1}}\Psi] \;=\; 0, \\ &\partial^i_{\dot{1}\dot{\alpha}_1\cdots\dot{\alpha}_{2f-1}}\partial^j_{\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{2f-1}}\Psi + \partial^j_{\dot{1}\dot{\beta}_1\cdots\dot{\beta}_{2f-1}}\partial^i_{\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{2f-1}}\Psi + \\ &\qquad\qquad + \; \{\partial^i_{\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{2f-1}}\Psi, \partial^j_{\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{2f-1}}\Psi\} \;=\; 0. \end{aligned} \tag{4.60}$$

Clearly, when all of the free $\dot{\alpha}$ and $\dot{\beta}$ indices in (4.60) are chosen to be one, we recover the $\mathcal{N}$-extended self-dual SYM equations in Leznov gauge (2.22).

Choosing, for instance, all of the $\dot{\alpha}$ indices equal to one and keeping the $\dot{\beta}$ indices arbitrary, we can interpret the equations (4.60) as equations on *symmetries*

$$\delta_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2b}}\Psi \;\equiv\; \partial_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2b}}\Psi \qquad \text{and} \qquad \delta^i_{\dot{\alpha}_1\cdots\dot{\alpha}_{2f}}\Psi \;\equiv\; \partial^i_{\dot{\alpha}_1\cdots\dot{\alpha}_{2f}}\Psi, \tag{4.61}$$

i.e.,

$$\begin{aligned} \partial_{\alpha\dot{1}\cdots\dot{1}}\delta_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{2b-1}}\Psi - \partial_{\alpha\dot{2}\dot{1}\cdots\dot{1}}\delta_{\beta\dot{1}\dot{\beta}_1\cdots\dot{\beta}_{2b-1}}\Psi \; + [\partial_{\alpha\dot{2}\dot{1}\cdots\dot{1}}\Psi, \delta_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{2b-1}}\Psi] \;&=\; 0, \\ \partial^i_{\dot{1}\cdots\dot{1}}\delta_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{2b-1}}\Psi - \partial^i_{\dot{2}\dot{1}\cdots\dot{1}}\delta_{\beta\dot{1}\dot{\beta}_1\cdots\dot{\beta}_{2b-1}}\Psi \; + [\partial^i_{\dot{2}\dot{1}\cdots\dot{1}}\Psi, \delta_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{2b-1}}\Psi] \;&=\; 0, \\ \partial^i_{\dot{1}\cdots\dot{1}}\delta^j_{\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{2f-1}}\Psi - \partial^i_{\dot{2}\dot{1}\cdots\dot{1}}\delta^j_{\dot{1}\dot{\beta}_1\cdots\dot{\beta}_{2f-1}}\Psi \; + \{\partial^i_{\dot{2}\dot{1}\cdots\dot{1}}\Psi, \delta^j_{\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{2f-1}}\Psi\} \;&=\; 0. \end{aligned} \tag{4.62}$$

In other words, the differential equations (4.61) describe graded Abelian flows on the space of solutions to (2.22). These flows are integral curves for the dynamical system (4.61). This system is the pendant – albeit in a particular gauge – to the system (3.46).

Finally, the field content in Leznov gauge is given by

$$
\begin{aligned}
\mathcal{A}_{\alpha\dot{1}\dot{\alpha}_1\cdots\dot{\alpha}_{2b-1}} &= \partial_{\alpha\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{2b-1}}\Psi,\\
\phi^i_{\dot{\alpha}_1\cdots\dot{\alpha}_{2f-2}} &= \tfrac{1}{2}\partial^i_{\dot{1}\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{2f-2}}\Psi,\\
W^{[ij]}_{(\dot{\alpha}_1\cdots\dot{\alpha}_{4f-2})} &= \tfrac{1}{2}\partial^{[i}_{\dot{2}(\dot{\alpha}_1\cdots\dot{\alpha}_{2f-1}}\partial^{j]}_{\dot{2}\dot{\alpha}_{2f}\cdots\dot{\alpha}_{4f-2})}\Psi,\\
\chi^{[ijk]}_{(\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{6f-3})} &= \tfrac{1}{2}\partial^{[i}_{\dot{2}(\dot{\alpha}_1\cdots\dot{\alpha}_{2f-1}}\partial^{j}_{\dot{2}\dot{\alpha}_{2f}\cdots\dot{\beta}_{4f-2}}\partial^{k]}_{\dot{2}\dot{\alpha}_{4f-1}\cdots\dot{\gamma}_{6f-3})}\Psi,\\
G^{[ijkl]}_{(\dot{2}\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{8f-4})} &= \tfrac{1}{2}\partial^{[i}_{\dot{2}(\dot{\alpha}_1\cdots\dot{\alpha}_{2f-1}}\partial^{j}_{\dot{2}\dot{\alpha}_{2f}\cdots\dot{\alpha}_{4f-2}}\partial^{k}_{\dot{2}\dot{\alpha}_{4f-1}\cdots\dot{\alpha}_{6f-3}}\partial^{l]}_{\dot{2}\dot{\alpha}_{6f-2}\cdots\dot{\alpha}_{8f-4})}\Psi.
\end{aligned}
\tag{4.63}
$$

Interestingly, the superfield expansion of the potential Ψ of the hierarchy does not involve the nonlocal operator $\partial^{-1}_{\alpha\dot{2}}$ which is due to equation (4.42).[38] This is contrary to the self-dual SYM case (cf. equations (2.23)).

5. Open topological B-model on the enhanced supertwistor space

The above chosen trivializiations of the holomorphic vector bundles $\mathcal{E}\to\mathcal{P}^{3|\mathcal{N}}$ and $\mathcal{E}\to\mathcal{P}^{3|\mathcal{N}}_{b,f}$ were convenient for the discussion of the self-dual SYM equations and the self-dual SYM hierarchies, respectively. It is well known, however, that there is a variety of other possible trivializations. In particular, it has been argued in [23] and explained in detail in [34] that the $\mathcal{N}$-extended self-dual SYM equations are gauge equivalent to hCS theory [42] on the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ by choosing certain non-holomorphic trivializations. In this section, we are going to extend this discussion and show that the moduli space $\mathcal{M}^{\mathcal{N}}_{\rm hCS}(b,f)$ of solutions to the equations of motion of hCS theory on the enhanced supertwistor space $\mathcal{P}^{3|\mathcal{N}}_{b,f}$ can bijectively be mapped onto the moduli space $\mathcal{M}^{\mathcal{N}}_{\rm SYMH}(b,f)$ of solutions to the $\mathcal{N}$-extended self-dual SYM hierarchy, i.e., we extend (4.17) according to

$$
\mathcal{M}_{\rm hol}(\mathcal{P}^{3|\mathcal{N}}_{b,f})\longleftrightarrow\ \mathcal{M}^{\mathcal{N}}_{\rm SYMH}(b,f)\ \longleftrightarrow\ \mathcal{M}^{\mathcal{N}}_{\rm hCS}(b,f). \tag{5.1}
$$

Furthermore, we then show that the open topological B-model describes the (truncated) hierarchies.

[38] For the definition of this operator using the Mandelstam-Leibbrandt prescription see, e.g., [59].

5.1. Holomorphic Chern-Simons theory on the enhanced supertwistor space

Consider a holomorphic rank n vector bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}_{b,f}$. Recall that on $\mathcal{P}^{3|\mathcal{N}}_{b,f}$ we have introduced the coordinates (4.2). Thus, the transition function $f = \{f_{+-}\}$ of $\mathcal{E}$ is annihilated by the vector fields

$$\bar{V}_1^{\pm} \;=\; \frac{\partial}{\partial \bar{z}^2_{\pm}}, \qquad \bar{V}_2^{\pm} \;=\; \frac{\partial}{\partial \bar{z}^1_{\pm}}, \qquad \partial_{\bar{\lambda}_{\pm}} \;=\; \frac{\partial}{\partial \bar{\lambda}_{\pm}} \qquad \text{and} \qquad \bar{\partial}^i_{\pm} \;=\; \frac{\partial}{\partial \bar{\eta}^{\pm}_i}, \tag{5.2}$$

which form a basis of vector fields of type $(0,1)$ on the (complexified) tangent bundle of the enhanced supertwistor space $\mathcal{P}^{3|\mathcal{N}}_{b,f}$.

Remember that the exterior derivative d on $\mathcal{P}^{3|\mathcal{N}}_{b,f}$ can always be decomposed into holomorphic and anti-holomorphic parts, i.e., $\mathrm{d} = \partial + \bar{\partial}$. Using (5.2), we obtain for the anti-holomorphic part

$$\bar{\partial} \;=\; \bar{\Theta}^{\alpha}_{\pm}\, \bar{V}^{\pm}_{\alpha} + \mathrm{d}\bar{\lambda}_{\pm}\, \partial_{\bar{\lambda}_{\pm}} + \mathrm{d}\bar{\eta}^{\pm}_i\, \bar{\partial}^i_{\pm},$$

where the $(0,1)$-forms are defined by

$$\bar{V}^{\pm}_{\beta} \lrcorner \bar{\Theta}^{\alpha}_{\pm} \;=\; \delta^{\alpha}_{\beta}, \qquad \partial_{\bar{\lambda}_{\pm}} \lrcorner \mathrm{d}\bar{\lambda}_{\pm} \;=\; 1 \qquad \text{and} \qquad \bar{\partial}^i_{\pm} \lrcorner \mathrm{d}\bar{\eta}^{\pm}_j \;=\; \delta^i_j$$

Let us now assume that the holomorphic vector bundle $\mathcal{E}$ is topologically trivial which implies that there exist some smooth matrix-valued functions $\hat{\psi} = \{\hat{\psi}_+, \hat{\psi}_-\}$ such that the transition function f_{+-} of $\mathcal{E}$ is given by

$$f_{+-} = \hat{\psi}_+^{-1} \hat{\psi}_-. \tag{5.3}$$

Therefore, we deduce from the condition $\bar{\partial} f_{+-} = 0$

$$(\bar{\partial} + \mathcal{A}^{0,1})\hat{\psi}_{\pm} \;=\; 0 \qquad \Longleftrightarrow \qquad \mathcal{A}^{0,1} \;=\; \hat{\psi}_{\pm} \bar{\partial} \hat{\psi}_{\pm}^{-1}, \tag{5.4}$$

what can concisely be rewritten as

$$\bar{\partial}_{\mathcal{A}} \hat{\psi}_{\pm} \;=\; 0, \tag{5.5}$$

where we have defined $\bar{\partial}_{\mathcal{A}} \equiv \bar{\partial} + \mathcal{A}^{0,1}$. Letting[39]

$$\mathcal{A}^{0,1} \;=\; \bar{\Theta}^{\alpha}_{\pm} \mathcal{A}^{\pm}_{\alpha} + \mathrm{d}\bar{\lambda}_{\pm} \mathcal{A}_{\bar{\lambda}_{\pm}} + \mathrm{d}\bar{\eta}^{\pm}_i \mathcal{A}^i_{\pm},$$

[39] In the sequel, we also assume that $\mathcal{A}^{0,1}$ does not contain anti-holomorphic fermionic components and depends holomorphically on $\eta^{\pm}_i$ [23,34,35].

we can rewrite the system (5.5) in components according to

$$(\bar{V}^+_\alpha + \mathcal{A}^+_\alpha)\hat{\psi}_\pm \;=\; 0, \tag{5.6a}$$

$$(\partial_{\bar{\lambda}_+} + \mathcal{A}_{\bar{\lambda}_+})\hat{\psi}_\pm \;=\; 0, \tag{5.6b}$$

$$(\bar{\partial}^i_+ + \mathcal{A}^i_+)\hat{\psi}_\pm \;=\; 0. \tag{5.6c}$$

Note that since the $(0,1)$-forms $\bar{\Theta}^\alpha_\pm$, $\mathrm{d}\bar{\lambda}_\pm$ and $\mathrm{d}\bar{\eta}^\pm_i$ are basis sections of the bundles $\overline{\mathcal{O}}(-2b_\alpha)$, $\overline{\mathcal{O}}(-2)$ and $\Pi\overline{\mathcal{O}}(2f_i)$, the components $\mathcal{A}^\pm_\alpha$, $\mathcal{A}_{\bar{\lambda}_\pm}$ and $\mathcal{A}^i_\pm$ must be sections of $\overline{\mathcal{O}}(-2b_\alpha)$, $\overline{\mathcal{O}}(-2)$ and $\Pi\overline{\mathcal{O}}(2f_i)$, respectively, such that $\mathcal{A}^{0,1}$ is $\mathbb{C}$-valued.[40]

Finally, the compatibility conditions of (5.5) are the equations of motion of hCS theory,

$$\bar{\partial}^2_{\mathcal{A}} \;=\; 0 \qquad \Longleftrightarrow \qquad \bar{\partial}\mathcal{A}^{0,1} + \mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1} \;=\; 0, \tag{5.7}$$

on the enhanced supertwistor space.

5.2. Bijection between moduli spaces

Above we have described hCS theory on the enhanced supertwistor space. In the following, we shall explain how the moduli space of solutions to the equations of motion of hCS theory is related to the moduli space of the $\mathcal{N}$-extended self-dual SYM hierarchy of the previous section.

Consider again the topologically trivial holomorphic vector bundle $\mathcal{E}\to\mathcal{P}^{3|\mathcal{N}}_{b,f}$ of the preceding subsection, which is characterized by the transition function $f=\{f_{+-}\}$, and furthermore the double fibration

$$\mathcal{P}^{3|\mathcal{N}}_{b,f} \quad \xleftarrow{\pi_2} \quad \mathcal{F}^{2(b_1+b_2)+3|2(f_1+\cdots+f_{\mathcal{N}})+\mathcal{N}} \quad \xrightarrow{\pi_1} \quad \mathbb{C}^{2(b_1+b_2+1)|2(f_1+\cdots+f_{\mathcal{N}})+\mathcal{N}}.$$

Let us now pull back $\mathcal{E}$ with the help of the projection π_2 to the correspondence space $\mathcal{F}^{2(b_1+b_2)+3|2(f_1+\cdots+f_{\mathcal{N}})+\mathcal{N}}$. By definition of the pull-back bundle, the pulled back transition functions are constant along the leaves

$$\mathcal{F}^{2(b_1+b_2)+3|2(f_1+\cdots+f_{\mathcal{N}})+\mathcal{N}}/\mathcal{P}^{3|\mathcal{N}}_{b,f} \;\cong\; \mathbb{C}^{2(b_1+b_2)|2(f_1+\cdots+f_{\mathcal{N}})}$$

of the fibration $\mathcal{F}^{2(b_1+b_2)+3|2(f_1+\cdots+f_{\mathcal{N}})+\mathcal{N}}\to\mathcal{P}^{3|\mathcal{N}}_{b,f}$. In other words, they are subject to the conditions

$$D^\pm_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha-1}}f_{+-} \;=\; 0 \;=\; D^i_{\pm\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1}}f_{+-}, \tag{5.8}$$

[40] Here, we again consider generic b_αs and f_is.

where the vector fields have been defined in (4.8) and, as before, we use the same letter f_{+-} for the pulled back transition function. Thus, we obtain from (5.5) together with (5.8) the following system of equations

$$(\pi_2^*\bar{\partial}_{\mathcal{A}})\hat{\psi}_\pm \;=\; 0, \quad D^+_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha-1}}\hat{\psi}_\pm \;=\; 0 \quad \text{and} \quad D^i_{+\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1}}\hat{\psi}_\pm \;=\; 0, \tag{5.9}$$

where the pulled back transition function is split according to $f_{+-} = \hat{\psi}_+^{-1}\hat{\psi}_-$. In addition, we also assume the existence of a gauge for the solutions to the equations of motion of the hCS theory in which the components $\mathcal{A}_{\bar{\lambda}_\pm}$ vanish identically. This is clearly equivalent to say that the bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}_{b,f}$ is holomorphically trivial on the rational curves $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}_{b,f}$. As the correspondence space is the direct product

$$\mathcal{F}^{2(b_1+b_2)+3|2(f_1+\cdots+f_\mathcal{N})+\mathcal{N}} \;=\; \mathbb{C}^{2(b_1+b_2+1)|2(f_1+\cdots+f_\mathcal{N})+\mathcal{N}} \times \mathbb{C}P^1,$$

this then implies that the pulled back bundle is holomorphically trivial ensuring the existence of a gauge where $\pi_2^*\mathcal{A}^{0,1} = 0$. Therefore, the system (5.9) can be gauge transformed to

$$\bar{\partial}\psi_\pm \;=\; 0, \tag{5.10a}$$

$$(D^+_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha-1}} + \mathcal{A}^+_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2b_\alpha-1}})\psi_\pm \;=\; 0, \tag{5.10b}$$

$$(D^i_{+\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1}} + \mathcal{A}^i_{+\dot{\alpha}_1\cdots\dot{\alpha}_{2f_i-1}})\psi_\pm \;=\; 0, \tag{5.10c}$$

which is nothing but the auxiliary linear system for the truncated $\mathcal{N}$-extended self-dual SYM hierarchy (4.11). Note that in (5.10a) $\bar{\partial}$ denotes the anti-holomorphic part of the exterior derivative on the correspondence space.

In summary, the moduli space of solutions to hCS theory defined on the enhanced supertwistor space can bijectively be mapped onto the moduli space of solutions to the $\mathcal{N}$-extended self-dual SYM hierarchy, i.e., we have established (5.1).

5.3. Self-dual super Yang-Mills hierarchies: An example

Let us now exemplify our discussion. Consider the truncated $\mathcal{N} = 2$ self-dual SYM hierarchy of type $(b_1, b_2, f_1, f_2) = (\frac{1}{2}, \frac{1}{2}, 1, 1)$. Its field equations are given by

$$\begin{aligned} \mathring{f}_{\dot{\alpha}\dot{\beta}} \;&=\; 0,\\ \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\mathring{\nabla}_{\alpha\dot{\alpha}}\mathring{\nabla}_{\beta\dot{\beta}}\mathring{\phi}^i \;&=\; 0,\\ \epsilon^{\dot{\alpha}\dot{\beta}}\mathring{\nabla}_{\alpha\dot{\alpha}}\mathring{W}^{[ij]}_{\dot{\beta}\dot{\gamma}} - 2\{\mathring{\phi}^{[i}, \mathring{\nabla}_{\alpha\dot{\gamma}}\mathring{\phi}^{j]}\} \;&=\; 0, \end{aligned} \tag{5.11}$$

and follow from (4.52). The $\mathcal{N}=2$ self-dual SYM equations, which are the first three equations of (2.17) (with $i,j=1,2$), are by construction a "subset" of (5.11). Namely, apply to the last equation of (5.11) the operator $\overset{\circ}{\nabla}_{\beta\dot\delta}$ and contract with $\epsilon^{\alpha\beta}$ to obtain

$$\epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\overset{\circ}{\nabla}_{\beta\dot\delta}\overset{\circ}{\nabla}_{\alpha\dot\alpha}\overset{\circ}{W}{}^{[ij]}_{\dot\beta\dot\gamma}-2\epsilon^{\alpha\beta}\{\overset{\circ}{\nabla}_{\beta\dot\delta}\overset{\circ}{\phi}{}^{[i},\overset{\circ}{\nabla}_{\alpha\dot\gamma}\overset{\circ}{\phi}{}^{j]}\} \;=\; 0,$$

where we have used the identity (4.57). Equations (4.43) together with (4.57) then imply

$$\begin{aligned}\overset{\circ}{f}_{\dot\alpha\dot\beta} \;&=\; 0,\\ \epsilon^{\alpha\beta}\overset{\circ}{\nabla}_{\alpha\dot\alpha}\overset{\circ}{\chi}{}^{i}_{\dot\beta\beta} \;&=\; 0,\\ \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\overset{\circ}{\nabla}_{\alpha\dot\alpha}\overset{\circ}{\nabla}_{\beta\dot\beta}\overset{\circ}{W}{}^{[ij]}_{\dot\gamma\dot\delta}-\epsilon^{\alpha\beta}\{\overset{\circ}{\chi}{}^{[i}_{\dot\gamma\alpha},\overset{\circ}{\chi}{}^{j]}_{\dot\delta\beta}\} \;&=\; 0,\end{aligned}\tag{5.12}$$

which reduce to the $\mathcal{N}=2$ self-dual SYM equations when the dotted indices of $\overset{\circ}{\chi}{}^{i}_{\dot\alpha\alpha}$ and $\overset{\circ}{W}{}^{[ij]}_{\dot\alpha\dot\beta}$ are chosen to be $\dot1$. Note that as i,j run only from 1 to 2, the last equation of (5.11) can be rewritten in the form

$$\begin{aligned}\overset{\circ}{f}_{\dot\alpha\dot\beta} \;&=\; 0,\\ \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\overset{\circ}{\nabla}_{\alpha\dot\alpha}\overset{\circ}{\nabla}_{\beta\dot\beta}\overset{\circ}{\phi}{}^{i} \;&=\; 0,\\ \epsilon^{\dot\alpha\dot\beta}\overset{\circ}{\nabla}_{\alpha\dot\alpha}\overset{\circ}{G}_{\dot\beta\dot\gamma}-\tfrac{3}{4}\epsilon_{ij}\{\overset{\circ}{\phi}{}^{i},\overset{\circ}{\nabla}_{\alpha\dot\gamma}\overset{\circ}{\phi}{}^{j}\} \;&=\; 0,\end{aligned}\tag{5.13}$$

where we have defined the anti-self-dual two-form according to $\overset{\circ}{G}_{\dot\alpha\dot\beta}\equiv\frac{3}{8}\epsilon_{ij}\overset{\circ}{W}{}^{[ij]}_{\dot\alpha\dot\beta}$.

On the other hand, hCS theory on the enhanced supertwistor space

$$\mathcal{P}^{3|2}[1,1|2,2],$$

where we again adopted the notation from the beginning of section 4, has already been considered in [35]. There, the analysis led to the same equations of motion. We note that $\mathcal{P}^{3|2}[1,1|2,2]$ is a Calabi-Yau supermanifold (see below). Therefore, we also have a well-defined action functional for the hCS theory and hence for the truncated self-dual SYM hierarchy,

$$S \;=\; \int \mathrm{d}^4x\ \mathrm{tr}\left\{\overset{\circ}{G}{}^{\dot\alpha\dot\beta}\overset{\circ}{f}_{\dot\alpha\dot\beta}+\tfrac{3}{8}\epsilon_{ij}\overset{\circ}{\phi}{}^{i}\,\overset{\circ}{\nabla}_{\alpha\dot\alpha}\overset{\circ}{\nabla}{}^{\alpha\dot\alpha}\overset{\circ}{\phi}{}^{j}\right\}\tag{5.14}$$

as well.

5.4. Hierarchies and open topological B-model

So far, we have been studying the (truncated) hierarchies of the $\mathcal{N}$-extended self-dual SYM theory from the field theoretic point of view. We will now argue that the open topological B-model describes certain corners of the hierarchies. This should be not that surprising, however, as we know since Witten's work [23] that the $\mathcal{N}=4$ self-dual SYM theory can be described by the open topological B-model defined on the supertwistor space. Remember that for having a well defined B-model, we need to impose the Calabi-Yau condition on the target manifold X, whose complex dimension is assumed to be $(3|\mathcal{N})$, in the sequel. The Calabi-Yau condition is reflected in the requirement of the vanishing of the first Chern number of the target space. Furthermore, we have seen in [42,23] that in this case the (cubic open string field theory of the) open topological B-model reduces to hCS theory. The action functional for the latter theory reads as

$$S \;=\; \int_Y \Omega\wedge \mathrm{tr}\left(\mathcal{A}^{0,1}\wedge\bar{\partial}\mathcal{A}^{0,1}+\tfrac{2}{3}\mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1}\right), \tag{5.15}$$

where Ω is the holomorphic measure on X, $\mathcal{A}^{0,1}$ represents the $(0,1|0,0)$-part of a super gauge potential $\mathcal{A}$ (that is, it is assumed that $\mathcal{A}^{0,1}$ does not contain anti-holomorphic fermionic components and does holomorphically depend on the fermionic coordinates) and "tr" denotes the trace over $\mathfrak{gl}(n,\mathbb{C})$. In (5.15), $Y\subset X$ is the submanifold of X which is obtained by the requirement that all of the complex conjugated fermionic coordinates are put to zero [23]. Let now the target space be the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$. Clearly, this only works for $\mathcal{N}=4$ since only then the supertwistor space is a Calabi-Yau manifold. In this case, the holomorphic measure Ω can be taken as

$$\Omega\big|_{\mathcal{U}_\pm} \;=\; \pm\,\mathrm{d}z^1_\pm\wedge\mathrm{d}z^2_\pm\wedge\mathrm{d}\lambda_\pm\mathrm{d}\eta_1^\pm\mathrm{d}\eta_2^\pm\mathrm{d}\eta_3^\pm\mathrm{d}\eta_4^\pm. \tag{5.16}$$

Since the equations of motion of hCS theory defined on $\mathcal{P}^{3|\mathcal{N}}$ are gauge equivalent to the $\mathcal{N}$-extended self-dual SYM equations on $\mathbb{R}^4$, $\mathcal{N}=4$ self-dual SYM theory is described by the open topological B-model. In sections 3 and 4, we have introduced a generalization of the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$, namely the enhanced supertwistor space $\mathcal{P}^{3|\mathcal{N}}_{b,f}$. The question one needs to address is to clarify when this space becomes a Calabi-Yau supermanifold.[41] The total first Chern number is given by

$$c_1(\mathcal{P}^{3|\mathcal{N}}_{b,f}) \;=\; 2(b_1+b_2)-2(f_1+\cdots+f_{\mathcal{N}})+2.$$

[41] For the discussion of super Ricci-flatness of Calabi-Yau supermanifolds, etc., see [38].

We therefore obtain a whole family of Calabi-Yau spaces which is parametrized by the discrete parameters b_α, f_i subject to

$$2(b_1 + b_2) - 2(f_1 + \cdots + f_{\mathcal{N}}) + 2 \ = \ 0.$$

Note that in this case the holomorphic measure is given by a similar looking expression as (5.16). Clearly, when we choose the parameters to be $b_\alpha = f_i = \frac{1}{2}$, we have $\mathcal{N} = 4$ supersymmetries and the hierarchies reduce to the standard $\mathcal{N} = 4$ self-dual SYM equations. In general, however, we can conclude that the open topological B-model defined on the manifold $\mathcal{P}^{3|\mathcal{N}}_{b,f}$ for fixed $\mathcal{N} > 0$, with the parameters b_α, f_i subject to the above condition, describes certain truncated versions of the $\mathcal{N}$-extended self-dual SYM hierarchy. Of course, for fixed $\mathcal{N}$ there is an infinite number of possibilities of choosing b_α and f_i. Note that one may also consider the formal limit $b_\alpha, f_i \to \infty$. This limit must be taken in such a way, however, that the Calabi-Yau condition $c_1 = 0$ holds. Thus, the B-model will also describe the full $\mathcal{N}$-extended self-dual SYM hierarchy.

6. Conclusions and outlook

In this paper, we described the construction of hidden symmetry algebras of the $\mathcal{N}$-extended self-dual SYM equations on $\mathbb{R}^4$ by means of the supertwistor correspondence. In particular, we exemplified our discussion by focusing on super Kac-Moody and Kac-Moody-Virasoro type symmetries. However, the presented algorithm does not only apply for those algebras but for any kind of algebra one may define on the supertwistor side, i.e., given some infinitesimal transformations of the transition function f_{+-} of some holomorphic vector bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ generated by some (infinite-dimensional) algebra, one can map those via the linearized Penrose-Ward transform to a corresponding set of symmetries on the gauge theory side. The only thing one should require for these transformations is that they preserve the complex structure on $\mathcal{P}^{3|\mathcal{N}}$.

Furthermore, the affine extension of the superconformal algebra led us to a new family of supertwistor spaces, which we denoted by $\mathcal{P}^{3|\mathcal{N}}_{b,f}$ and called enhanced supertwistor spaces. The supertwistor correspondence for those spaces eventually gave us the $\mathcal{N}$-extended self-dual SYM hierarchies, which in turn describe graded Abelian symmetries of the $\mathcal{N}$-extended self-dual SYM equations. We have shown that the moduli spaces of solutions to those hierarchies can bijectively be mapped to the moduli spaces of solutions to hCS theory on $\mathcal{P}^{3|\mathcal{N}}_{b,f}$. As $\mathcal{P}^{3|\mathcal{N}}_{b,f}$ turned out to be a Calabi-Yau supermanifold for certain

values of b_α and f_i, we have also seen that the open topological B-model on $\mathcal{P}^{3|\mathcal{N}}_{b,f}$ describes those (truncated) hierarchies.

There are a lot of open issues which certainly deserve further investigations:

• The first task is to generalize the above discussion to the full $\mathcal{N} = 4$ SYM theory. In restricting to the self-dual truncation, we have just given the first step. It is well known that $\mathcal{N} = 3$ (4) SYM theory in four dimensions is related via the supertwistor correspondence to a quadric of dimension (5|6) which lives in $\mathcal{P}^{3|3} \times \mathcal{P}^{3|3}$ (see [25,23,34] and references therein). The latter space and hence the quadric need to be covered with at least four coordinate patches. Hence, due to this four-set (open) covering certain technicalities in constructing the symmetry algebras will appear, but nevertheless there will be no principal problems. One simply needs to work out the details.

• Having given the supertwistor construction of hidden symmetries of the full $\mathcal{N} = 3$ (4) SYM theory, one needs to investigate – using, e.g., twistorial methods – the quantum corrections of the obtained symmetry algebras. For instance, by passing to the quantum regime, one will not obtain centerless Kac-Moody algebras such as (3.31) and (3.56), but rather

$$[\delta^m_a, \delta^n_b\} \;=\; f_{ab}{}^c\delta^{m+n}_c + k\delta_{ab}\delta_{m,-n},$$

where k is the level of the Kac-Moody algebra. Such algebras can in turn be used to define representations of so-called quantum (super) Yangians (see, e.g., [60] for the case of two-dimensional CFTs and [46,61] for the case of quantum SDYM theory).[42] Such quantum Yangians play an increasing and important role in the investigation of quantum integrability of the superconformal gauge theory [21].

• Another issue also worthwhile to explore is the construction of hidden symmetry algebras (and hierarchies) of gravity theories, in particular of conformal supergravity (see, e.g., [63] for a review). The description of conformal supergravity in terms of the supertwistor correspondence has been discussed in [23,29,37]. By applying similar techniques as those presented in this paper, one will eventually obtain infinite-dimensional symmetry algebras (and correspondingly an infinite number of conserved nonlocal charges) of the conformal supergravity equations, generalizing the results known for self-dual gravity (see, e.g., references [64]).

[42] For the definition of (super) Yangians see, e.g., references [14-16,21,62].

• Finally, we address another point for further investigations. It has been conjectured by Ward [65] that all integrable models in $D < 4$ dimensions can be obtained from the self-dual YM equations in four dimensions. Typical examples of such systems are the nonlinear Schrödinger equation, the Korteweg-de Vries equation, the sine-Gordon model, etc. All of them follow from the self-dual Yang-Mills equations by incorporating suitable algebraic ansätze for the self-dual gauge potential followed by a dimensional reduction. In a similar spirit, the Ward conjecture can be supersymmetrically extended in order to derive the supersymmetric versions of the above-mentioned models. Therefore, it would be of interest to take the $\mathcal{N}$-extended self-dual SYM hierarchy presented in this paper and to derive the corresponding super hierarchies of these integrable systems in $D < 4$ dimensions and to compare them with results known in the literature.

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Appendix A. Hidden symmetries and sheaf cohomology

In this appendix, we shall present a more formal approach to the symmetry algebras by using the sheaf cohomology machinery. In particular, we talk about super Kac-Moody and super Virasoro symmetries.

A.1. Čech description of holomorphic vector bundles

First, let us recall some basic definitions. Consider a complex (super)manifold M with the (open) covering $\mathfrak{U} = \{\mathcal{U}_m\}$. Furthermore, we are interested in smooth maps from open subsets of M into the non-Abelian group $GL(n,\mathbb{C})$ as well as in a sheaf $\mathfrak{G}$ of such $GL(n,\mathbb{C})$-valued functions. A q-cochain of the covering $\mathfrak{U}$ with values in $\mathfrak{G}$ is a collection $\psi = \{\psi_{m_0\cdots m_q}\}$ of sections of the sheaf $\mathfrak{G}$ over nonempty intersections $\mathcal{U}_{m_0} \cap \cdots \cap \mathcal{U}_{m_q}$. We

will denote the set of such q-cochains by $C^q(\mathfrak{U},\mathfrak{G})$. We stress that it has a group structure, where the multiplication is just pointwise multiplication.

We may define the subsets of cocylces $Z^q(\mathfrak{U},\mathfrak{G}) \subset C^q(\mathfrak{U},\mathfrak{G})$. For example, for $q=0,1$ these are given by

$$\begin{aligned} Z^0(\mathfrak{U},\mathfrak{G}) &\equiv \{\psi \in C^0(\mathfrak{U},\mathfrak{G}) \mid \psi_m = \psi_n \ \text{ on } \ \mathcal{U}_m \cap \mathcal{U}_n \neq \emptyset\}, \\ Z^1(\mathfrak{U},\mathfrak{G}) &\equiv \{\psi \in C^1(\mathfrak{U},\mathfrak{G}) \mid \psi_{nm} = \psi_{mn}^{-1} \ \text{ on } \ \mathcal{U}_m \cap \mathcal{U}_n \neq \emptyset \\ &\qquad\qquad \text{and } \ \psi_{mn}\psi_{np}\psi_{pm} = 1 \ \text{ on } \ \mathcal{U}_m \cap \mathcal{U}_n \cap \mathcal{U}_p \neq \emptyset\}. \end{aligned}$$

These sets will be of particular interest later on. We remark that from the first of these two definitions it follows that $Z^0(\mathfrak{U},\mathfrak{G})$ coincides with the group

$$H^0(M,\mathfrak{G}) \ \equiv \ \mathfrak{G}(M) \ = \ \Gamma(M,\mathfrak{G}),$$

which is the group of global sections of the sheaf $\mathfrak{G}$. Note that in general the subset $Z^1(\mathfrak{U},\mathfrak{G}) \subset C^1(\mathfrak{U},\mathfrak{G})$ is not a subgroup of the group $C^1(\mathfrak{U},\mathfrak{G})$.

We say that two cocycles $f, f' \in Z^1(\mathfrak{U},\mathfrak{G})$ are equivalent if $f'_{mn} = \psi_m^{-1} f_{mn} \psi_n$ for some $\psi \in C^0(\mathfrak{U},\mathfrak{G})$. The set of equivalence classes induced by this equivalence relation is the first cohomology set and denoted by $H^1(\mathfrak{U},\mathfrak{G})$. If the $\mathcal{U}_m$ are all Stein manifolds we have the bijection

$$H^1(\mathfrak{U},\mathfrak{G}) \ \cong \ H^1(M,\mathfrak{G}).$$

Furthermore, we shall also need the sheaf of holomorphic sections of the trivial bundle $M \times GL(n,\mathbb{C})$, which we denote by $\mathcal{O}_{GL}$, in the sequel.

Let us now stick to the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ and the correspondence space $\mathcal{F}^{5|2\mathcal{N}}$ with their two-set open coverings $\mathfrak{U} = \{\mathcal{U}_+, \mathcal{U}_-\}$ and $\tilde{\mathfrak{U}} = \{\tilde{\mathcal{U}}_+, \tilde{\mathcal{U}}_-\}$, respectively. We note that $\mathcal{U}_\pm$ and $\tilde{\mathcal{U}}_\pm$ are indeed Stein manifolds. Then any cocyle $f = \{f_{+-}\} \in Z^1(\mathfrak{U},\mathfrak{G})$ defines uniquely a complex rank n vector bundle $\mathcal{E}$ over $\mathcal{P}^{3|\mathcal{N}}$ and via π_2 over $\mathcal{F}^{5|2\mathcal{N}}$. Equivalent cocyles define isomorphic complex vector bundles and hence, since $\mathcal{U}_\pm$ are Stein manifolds, the isomorphism class of complex rank n vector bundles is parametrized by $H^1(\mathcal{P}^{3|\mathcal{N}},\mathfrak{G})$. Choosing $\mathfrak{G}$ to be $\mathcal{O}_{GL}$, we see that holomorphic vector bundles over the supertwistor space are parametrized by $H^1(\mathcal{P}^{3|\mathcal{N}},\mathcal{O}_{GL})$, i.e., $\mathcal{M}_{\rm hol}(\mathcal{P}^{3|\mathcal{N}}) \subset H^1(\mathcal{P}^{3|\mathcal{N}},\mathcal{O}_{GL})$.

A.2. Super Kac-Moody symmetries

Let us study small perturbations of the transition function f_{+-} of a holomorphic vector bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ and its pull-back bundle $\pi_2^* \mathcal{E} \to \mathcal{F}^{5|2\mathcal{N}}$. Consider $\mathcal{F}^{5|2\mathcal{N}}$ with the covering $\tilde{\mathfrak{U}} = \{\tilde{\mathcal{U}}_+, \tilde{\mathcal{U}}_-\}$. Then we define an action of $C^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL})$ on $Z^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL})$ by

$$h \,:\, f_{+-} \;\mapsto\; h_{+-} f_{+-} h_{-+}^{-1}, \tag{A.1}$$

where $h = \{h_{+-}, h_{-+}\} \in C^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL})$ and $f = \{f_{+-}\} \in Z^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL})$. Obviously, the group $C^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL})$ acts transitively on $Z^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL})$, since for an arbitrary Čech 1-cocycle $f \in Z^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL})$ one can find an $h \in C^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL})$ such that $f_{+-} = h_{+-} h_{-+}^{-1}$ and $f_{-+} = h_{-+} h_{+-}^{-1}$. The stabilizer of the trivial Čech 1-cocycle, given by $f = 1$, is simply the subgroup

$$C^1_\Delta(\tilde{\mathfrak{U}}, \mathcal{O}_{GL}) \;\equiv\; \{h \in C^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL}) \,|\, h_{+-} = h_{-+}\}$$

implying that $Z^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL})$ can be identified with the coset

$$Z^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL}) \;\cong\; C^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL}) / C^1_\Delta(\tilde{\mathfrak{U}}, \mathcal{O}_{GL}).$$

Now we are interested in infinitesimal deformations of the transition function f_{+-}. Consider the sheaf $\mathcal{O}_{\mathfrak{gl}}$ of holomorphic sections of the bundle $\mathcal{F}^{5|2\mathcal{N}} \times \mathfrak{gl}(n, \mathbb{C})$ and the sheaves $\mathcal{S}_{GL}$ and $\mathcal{S}_{\mathfrak{gl}}$ consisting of those $GL(n, \mathbb{C})$-valued and $\mathfrak{gl}(n, \mathbb{C})$-valued smooth functions, respectively, that are annihilated by $\partial_{\bar{\lambda}_\pm}$. We have a natural infinitesimal action of the group $C^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL})$ on the space $Z^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL})$ which is induced by the linearization of (A.1). Namely, we get

$$\delta f_{+-} \;=\; \delta h_{+-} f_{+-} - f_{+-} \delta h_{-+}, \tag{A.2}$$

where $\delta h = \{\delta h_{+-}, \delta h_{-+}\} \in C^1(\tilde{\mathfrak{U}}, \mathcal{O}_{\mathfrak{gl}})$ and $f = \{f_{+-}\} \in Z^1(\tilde{\mathfrak{U}}, \mathcal{O}_{GL})$. As in subsection 3.1, we introduce the $\mathfrak{gl}(n, \mathbb{C})$-valued function

$$\varphi_{+-} \;\equiv\; \psi_+ (\delta f_{+-}) \psi_-^{-1}, \tag{A.3}$$

where $\psi_\pm \in C^0(\tilde{\mathfrak{U}}, \mathcal{S}_{GL})$ and $f_{+-} = \psi_+^{-1} \psi_-$, as before. Here, $C^0(\tilde{\mathfrak{U}}, \mathcal{S}_{GL})$ is the group of 0-cochains. From equation (A.2) it is then immediate that

$$\varphi_{+-} \;=\; -\varphi_{-+}.$$

Moreover, φ_{+-} is annihilated by $\partial_{\bar{\lambda}_\pm}$. Thus, it defines an element $\varphi \in Z^1(\tilde{\mathfrak{U}}, \mathcal{S}_{\mathfrak{gl}})$ with $\varphi = \{\varphi_{+-}\}$. Any 1-cocycle with values in $\mathcal{S}_{\mathfrak{gl}}$ is a 1-coboundary since $H^1(\mathcal{F}^{5|2\mathcal{N}}, \mathcal{S}_{\mathfrak{gl}}) = 0$. Therefore, we have

$$\varphi_{+-} \ = \ \phi_+ - \phi_-, \tag{A.4}$$

where $\phi = \{\phi_+, \phi_-\} \in C^0(\tilde{\mathfrak{U}}, \mathcal{S}_{\mathfrak{gl}})$.

Linearizing $f_{+-} = \psi_+^{-1}\psi_-$, we obtain

$$\delta f_{+-} \ = \ f_{+-}\psi_-^{-1}\delta\psi_- - \psi_+^{-1}\delta\psi_+ f_{+-}.$$

and hence, by virtue of the equations (A.3) and (A.4) we obtain

$$\delta\psi_\pm \ = \ -\phi_\pm\psi_\pm. \tag{A.5}$$

In summary, given some $\delta h = \{\delta h_{+-}, \delta h_{-+}\}$ one derives via (A.2)-(A.4) the perturbations $\delta\psi_\pm$.[43] Now one may proceed as in subsection 3.1 to arrive at the formulas (3.12) for the nontrivial symmetries. Symmetries obtained this way are called super Kac-Moody symmetries.

A.3. Super Virasoro symmetries

Above we have introduced Kac-Moody symmetries of the $\mathcal{N}$-extended super SDYM equations which were generated by the algebra $C^1(\tilde{\mathfrak{U}}, \mathcal{O}_{\mathfrak{gl}})$. In this subsection we focus on symmetries, which are related with the group of local biholomorphisms of the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$.

Consider the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$. Remember that it comes with the local coordinates $(Z^a_\pm) = (z^\alpha_\pm, \lambda^\pm_{\dot{\alpha}}, \eta^\pm_i)$. On the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$, they are related by the transition function t^a_{+-}, i.e.,

$$Z^a_+ \ = \ t^a_{+-}(Z^b_-).$$

Let us denote the group of local biholomorphisms of the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ by $\mathfrak{H}_{\mathcal{P}}$, which is the subgroup of the diffeomorphism group of $\mathcal{P}^{3|\mathcal{N}}$ which consists of those maps from $\mathcal{P}^{3|\mathcal{N}} \to \mathcal{P}^{3|\mathcal{N}}$ which preserve the complex structure of $\mathcal{P}^{3|\mathcal{N}}$. To $\mathfrak{H}_{\mathcal{P}}$ corresponds the algebra

$$C^0(\mathfrak{U}, \mathcal{V}_{\mathcal{P}})$$

[43] In our example of subsection 3.2, δh was given by $\delta h_{+-} = \delta h_{-+} = \lambda_+^m X_a$.

of 0-cochains of $\mathcal{P}^{3|\mathcal{N}}$ with values in the sheaf $\mathcal{V}_\mathcal{P}$ of holomorphic vector fields on the supertwistor space. Let us now consider the algebra $C^1(\mathfrak{U},\mathcal{V}_\mathcal{P})$ whose elements are collections of vector fields

$$\chi \;=\; \{\chi_{+-},\chi_{-+}\} \;=\; \left\{\chi^a_{+-}\partial^+_a,\chi^a_{-+}\partial^-_a\right\}, \tag{A.6}$$

where we have abbreviated $\partial^\pm_a \equiv \partial/\partial Z^a_\pm$. In particular, the χ_{+-} and χ_{-+} are elements of the algebra $\mathcal{V}_\mathcal{P}(\mathcal{U}_+\cap\mathcal{U}_-)$ of holomorphic vector fields on the intersection $\mathcal{U}_+\cap\mathcal{U}_-$. Thus, $C^1(\mathfrak{U},\mathcal{V}_\mathcal{P})$ can be decomposed according to

$$C^1(\mathfrak{U},\mathcal{V}_\mathcal{P}) \;\cong\; \mathcal{V}_\mathcal{P}(\mathcal{U}_+\cap\mathcal{U}_-)\oplus\mathcal{V}_\mathcal{P}(\mathcal{U}_+\cap\mathcal{U}_-).$$

Kodaira-Spencer deformation theory then tells us that the algebra $C^1(\mathfrak{U},\mathcal{V}_\mathcal{P})$ acts on the transition functions t^a_{+-} according to

$$\delta t^a_{+-} \;=\; \chi^a_{+-} - \chi^b_{-+}\partial^-_b t^a_{+-} \tag{A.7}$$

which can equivalently be rewritten as

$$\delta t_{+-} \;\equiv\; \delta t^a_{+-}\partial^+_a \;=\; \chi_{+-}-\chi_{-+}. \tag{A.8}$$

Let us consider a subalgebra

$$C^1_\Delta(\mathfrak{U},\mathcal{V}_\mathcal{P}) \;\equiv\; \{\chi\in C^1(\mathfrak{U},\mathcal{V}_\mathcal{P})\,|\,\chi_{+-}=\chi_{-+}\}$$

of the algebra $C^1(\mathfrak{U},\mathcal{V}_\mathcal{P})$. Then the space

$$Z^1(\mathfrak{U},\mathcal{V}_\mathcal{P}) \;=\; \{\chi\in C^1(\mathfrak{U},\mathcal{V}_\mathcal{P})\,|\,\chi_{+-}=-\chi_{-+}\}$$

is given by the quotient

$$Z^1(\mathfrak{U},\mathcal{V}_\mathcal{P}) \;\equiv\; C^1(\mathfrak{U},\mathcal{V}_\mathcal{P})/C^1_\Delta(\mathfrak{U},\mathcal{V}_\mathcal{P}).$$

We stress that the transformations (A.7) change the complex structure of $\mathcal{P}^{3|\mathcal{N}}$ if $\chi_{+-}\neq\chi_+-\chi_-$, where $\{\chi_+,\chi_-\}\in C^0(\mathfrak{U},\mathcal{V}_\mathcal{P})$. Therefore,

$$H^1(\mathfrak{U},\mathcal{V}_\mathcal{P}) \;=\; Z^1(\mathfrak{U},\mathcal{V}_\mathcal{P})/C^0(\mathfrak{U},\mathcal{V}_\mathcal{P})$$

is the tangent space (at a chosen complex structure) of the moduli space of deformations of the complex structure on $\mathcal{P}^{3|\mathcal{N}}$.

Consider now a holomorphic vector bundle $\mathcal{E}$ over the supertwistor space. We may define the following holomorphic action (thus preserving the complex structure on $\mathcal{P}^{3|\mathcal{N}}$) of the algebra $C^0(\mathfrak{U}, \mathcal{V}_{\mathcal{P}})$ on the transition functions f_{+-} of the holomorphic vector bundle $\mathcal{E}$:

$$\delta f_{+-} \;=\; \chi_+(f_{+-}) - \chi_-(f_{+-}), \tag{A.9}$$

with χ_+ and χ_- restricted to $\mathcal{U}_+ \cap \mathcal{U}_-$. In a similar manner, $C^0(\mathfrak{U}, \mathcal{V}_{\mathcal{F}})$ acts on the pulled-back transition functions of the pull-back bundle $\pi_2^*\mathcal{E} \to \mathcal{F}^{5|3\mathcal{N}}$. Here, $\mathcal{V}_{\mathcal{F}}$ is the sheaf of holomorphic vector fields on the correspondence space $\mathcal{F}^{5|3\mathcal{N}}$.

After this little digression, we may now follow the lines presented in subsection 3.1 to arrive at the formulas (3.12) for the nontrivial symmetries. We refer to symmetries obtained in this way as super Virasoro symmetries since they are related with the group of local biholomorphisms.

Appendix B. Remarks on almost complex structures

Consider the superspace $(\mathbb{R}^{2m|2n}, I)$, where I is the canonical metric on $\mathbb{R}^{2m|2n}$,

$$I \;\equiv\; \begin{pmatrix} \mathbf{1}_{2m} & 0 \\ 0 & \omega_{2n} \end{pmatrix} \qquad \text{and} \qquad \omega_{2n} \;\equiv\; \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}. \tag{B.1}$$

The group of isometries of $\mathbb{R}^{2m|2n}$ is the matrix supergroup

$$OSp(2m|2n) \;\subset\; GL(2m|2n),$$

where $GL(2m|2n)$ is the group of all invertible matrices of dimension $(4m^2+4n^2|8mn)$. In particular,

$$OSp(2m|2n) \;\equiv\; \{g \in GL(2m|2n) \mid {}^{\mathrm{st}}g\, I\, g \;=\; I\}. \tag{B.2}$$

The corresponding Lie superalgebra $\mathfrak{osp}(2m|2n)$ is given by

$$\mathfrak{osp}(2m|2n) \;=\; \{X \in \mathfrak{gl}(2m|2n) \mid {}^{\mathrm{st}}X\, I + X\, I \;=\; 0\}. \tag{B.3}$$

In (B.2) and (B.3), the superscript “st” denotes the supertranspose which is defined as

$$X \;=\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \Longrightarrow \qquad {}^{\mathrm{st}}X \;\equiv\; \begin{pmatrix} {}^{\mathrm{t}}A & -\,{}^{\mathrm{t}}C \\ {}^{\mathrm{t}}B & {}^{\mathrm{t}}D \end{pmatrix}. \tag{B.4}$$

Here, A and D are even $2m \times 2m$ and $2n \times 2n$ matrices, while B and C are odd $2m \times 2n$ and $2n \times 2m$ matrices, respectively. Explicitly, (B.3) reads as

$$
{}^{\mathrm{t}}\!A + A \;=\; 0, \qquad B - {}^{\mathrm{t}}C\omega_{2m} \;=\; 0 \qquad \text{and} \qquad {}^{\mathrm{t}}D\omega_{2m} + \omega_{2m}D \;=\; 0, \tag{B.5}
$$

where X is given by (B.4).

An almost complex structure on the space $\mathbb{R}^{2m|2n}$ is an endomorphism $\mathcal{J}$ from $\mathbb{R}^{2m|2n} \to \mathbb{R}^{2m|2n}$ with $\mathcal{J}^2 = -\mathbf{1}$. In matrix representation we have

$$
\mathcal{J} \;=\; \begin{pmatrix} J_{2m} & 0 \\ 0 & J_{2n} \end{pmatrix}, \qquad \text{with} \qquad J_{2k} \;\equiv\; \begin{pmatrix} 0 & -\mathbf{1}_k \\ \mathbf{1}_k & 0 \end{pmatrix}. \tag{B.6}
$$

As we are interested in the moduli space of all (constant) almost complex structures, we need the subgroup $H \subset OSp_{\mathrm{c}}(2m|2n)$ (the subscript "c" stands for the connected component of $OSp(2m|2n)$, i.e., $\mathrm{sdet}(g) = +1$ for $g \in OSp(2m|2n)$) such that

$$
H \;\equiv\; \{g \in OSp_{\mathrm{c}}(2m|2n) \mid g^{-1}\,\mathcal{J}\,g \;=\; \mathcal{J}\}. \tag{B.7}
$$

The corresponding algebra is

$$
\mathfrak{h} \;=\; \{X \in \mathfrak{osp}(2m|2n) \mid [X, \mathcal{J}] \;=\; 0\}. \tag{B.8}
$$

Writing out explicitly the condition $[X, \mathcal{J}] = 0$ together with the explicit form of X given in (B.4), one realizes that $A \in \mathfrak{so}(2m) \cap \mathfrak{sp}(2m, \mathbb{R})$, that is, $A \in \mathfrak{u}(m)$. Similarly, both ${}^{\mathrm{t}}D\omega_{2n} + \omega_{2n}D = 0$ and $[X, \mathcal{J}] = 0$ show that $D \in \mathfrak{so}(2n) \cap \mathfrak{sp}(2n, \mathbb{R}) \cong \mathfrak{u}(n)$. The remaining condition given in (B.5) together with $[X, \mathcal{J}] = 0$ yield that B and C are of the form

$$
B \;=\; \begin{pmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{pmatrix} \qquad \text{and} \qquad C \;=\; -\begin{pmatrix} {}^{\mathrm{t}}B_2 & {}^{\mathrm{t}}B_1 \\ -\,{}^{\mathrm{t}}B_1 & {}^{\mathrm{t}}B_2 \end{pmatrix}, \tag{B.9}
$$

where the $B_{1,2}$ are real $m \times n$-matrices. Then, we may identify B and C according to

$$
B \;\mapsto\; \mathrm{i}(B_1 + \mathrm{i}B_2) \qquad \text{and} \qquad C \;\mapsto\; {}^{\mathrm{t}}B_2 + \mathrm{i}\,{}^{\mathrm{t}}B_1, \tag{B.10}
$$

i.e., C is the negative Hermitian adjoint of B. In summary, the algebra $\mathfrak{h}$ can therefore be identified with $\mathfrak{u}(m|n)$.[44] Hence, the supercoset

$$
OSp_{\mathrm{c}}(2m|2n)/U(m|n) \tag{B.11}
$$

[44] Recall, the matrices $X \in \mathfrak{u}(m|n)$ are defined by the condition that A and D are skew-Hermitian while C is the negative Hermitian adjoint of B (cf., e.g., [21]).

parametrizes all almost complex structures on $\mathbb{R}^{2m|2n}$. Note that as the (real) dimension of $OSp_{\rm c}(2m|2n)$ is $(m(2m-1)+n(2n+1)|4mn)$ and of $U(m|n)$ is $(m^2+n^2|2mn)$, respectively, the dimension of the supercoset (B.11) is $(m(m-1)+n(n+1)|2mn)$.

Let us now stick to the case when $m=2$ and $n=\mathcal{N}$. As in the purely bosonic case, we may introduce

$$\mathcal{P}_{\mathcal{N}} \;\equiv\; P(\mathbb{R}^{4|2\mathcal{N}}, OSp_{\rm c}(4|2\mathcal{N})) \times_{OSp_{\rm c}(4|2\mathcal{N})} (OSp_{\rm c}(4|2\mathcal{N})/U(2|\mathcal{N})), \tag{B.12}$$

where $P(\mathbb{R}^{4|2\mathcal{N}}, OSp_{\rm c}(4|2\mathcal{N}))$ is the principal $OSp_{\rm c}(4|2\mathcal{N})$-frame bundle on $\mathbb{R}^{4|2\mathcal{N}}$. Since the supermanifold $\mathbb{R}^{4|2\mathcal{N}}$ is trivial, (B.12) becomes

$$\mathcal{P}_{\mathcal{N}} \;=\; \mathbb{R}^{4|2\mathcal{N}} \times (OSp_{\rm c}(4|2\mathcal{N})/U(2|\mathcal{N})). \tag{B.13}$$

Clearly, when $\mathcal{N}=0$ (B.13) reduces to $\mathbb{R}^4 \times \mathbb{C}P^1$, which is diffeomorphic to the bosonic twistor space $\mathcal{P}^3 = \mathbb{C}P^3 \backslash \mathbb{C}P^1$. Therefore, one may regard (B.13) as another super extension of the twistor space.

Appendix C. Superconformal algebra

In this appendix, we shall give the (anti)commutation relations of the generators (3.32) of the superconformal algebra. They are

$$\begin{aligned}
[P_{\alpha\dot\alpha}, K_{\beta\dot\beta}] &= 2(\epsilon_{\alpha\beta} J_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta} J_{\alpha\beta}) - \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} D,\\
\{Q_{i\alpha}, Q^j_{\dot\alpha}\} &= \delta^j_i P_{\alpha\dot\alpha}, \qquad \{K^{i\alpha}, K^{\dot\alpha}_j\} \;=\; -\delta^i_j K^{\alpha\dot\alpha},\\
\{Q_{i\alpha}, K^{j\beta}\} &= 2\delta^j_i (J_\alpha{}^\beta + \tfrac{1}{4}\delta^\beta_\alpha D) + \tfrac{1}{2}\delta^\beta_\alpha \delta^j_i (1 - \tfrac{4}{\mathcal{N}})A - \delta^\beta_\alpha T^j_i,\\
[T^i_j, K^{k\alpha}] &= -(\delta^k_j K^{i\alpha} - \tfrac{1}{\mathcal{N}}\delta^i_j K^{k\alpha}),\\
[A, K^{i\alpha}] &= -\tfrac{1}{2} K^{i\alpha}, \qquad [D, K^{i\alpha}] \;=\; -\tfrac{1}{2}K^{i\alpha},\\
[J_{\alpha\beta}, K^{i\gamma}] &= \tfrac{1}{2}\epsilon_{\delta(\alpha}\delta^\gamma_{\beta)} K^{i\delta}, \qquad [P_{\alpha\dot\alpha}, K^{i\beta}] \;=\; -\delta^\beta_\alpha Q^i_{\dot\alpha}\\
[T^j_i, Q_{k\alpha}] &= \delta^j_k Q_{i\alpha} - \tfrac{1}{\mathcal{N}}\delta^j_i Q_{k\alpha},\\
[A, Q_{i\alpha}] &= \tfrac{1}{2} Q_{i\alpha}, \qquad [D, Q_{i\alpha}] \;=\; \tfrac{1}{2} Q_{i\alpha},\\
[J_{\alpha\beta}, Q_{i\gamma}] &= -\tfrac{1}{2}\epsilon_{\gamma(\alpha} Q_{i\beta)}, \qquad [Q_{i\beta}, K^{\alpha\dot\alpha}] \;=\; \delta^\beta_\alpha K^{\dot\alpha}_i,\\
[T^j_i, T^l_k] &= \delta^j_k T^l_i - \delta^l_i T^j_k,\\
[D, P_{\alpha\dot\alpha}] &= P_{\alpha\dot\alpha}, \qquad [D, K^{\alpha\dot\alpha}] = -K^{\alpha\dot\alpha},\\
[J_{\alpha\beta}, K^{\gamma\dot\gamma}] &= \tfrac{1}{2}\epsilon_{\delta(\alpha}\delta^\gamma_{\beta)} K^{\delta\dot\gamma}, \qquad [J_{\alpha\beta}, P_{\gamma\dot\gamma}] \;=\; -\tfrac{1}{2}\epsilon_{\gamma(\alpha} P_{\beta)\dot\gamma},\\
[J_{\alpha\beta}, J^{\gamma\delta}] &= -\delta^{(\gamma}_{(\alpha} J_{\beta)}{}^{\delta)}.
\end{aligned} \tag{C.1}$$

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